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Banach fixed point theorem in a Cone pentagonal metric spaces

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Abstract

In this paper, we prove Banach fixed point theorem in cone pentagonal metric spaces without assuming the normality condition. Our results improve and extend recent known results.

Keywords: Cone metric space, fixed points, contraction mapping principle, ordered Banach space.
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1. Introduction

In 2007, Long-Guang and Xian [6] introduced the concept of a cone metric space, they replaced set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [1, 5, 8]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Recently, Azam et al. [3] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a cone rectangular metric space setting. In 2012, Rashwan and Saleh [7] improve and extended the result of Azam et al. [3] by removing the normality condition.

Very recently, Garg and Agarwal [4] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a cone pentagonal metric space setting using the normality condition.

Motivated by these results of [4, 7], it is our purpose in this paper to continue the study of fixed point theorem in cone pentagonal metric space setting. Our results improve and extend the results of [4, 7].

2. Preliminaries

We present some definitions introduced in [2, 3, 4, 6, 7], which will be needed in the sequel.

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Definition 2.1. Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$.
(2) $a, b \in \mathbb{R}, \;\; a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$.
(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Definition 2.2. Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 2.3. A cone P is called normal if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \tag{1}$$

The least positive number k satisfying (1) is called the normal constant of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4. Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.
(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

Remark 2.5. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [6]).

Definition 2.6. Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.
(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
(3) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [Rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 2.7. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [3]).

Definition 2.8. Let X be a non empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for $x, y \in X$.
(3) $d(x, y) = d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u, \in X - \{x, y\}$ [Pentagonal property].

Then d is called a cone Pentagonal metric on X, and (X, d) is called a cone Pentagonal metric space.

Remark 2.9. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [4]).

Definition 2.10. Let (X, d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in N$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.11. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is called Cauchy sequence in (X, d).

Definition 2.12. If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone pentagonal metric space.

Definition 2.13. Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi : P \to P$ satisfying:

(1) $0 < \varphi(t) < t$ for all $t \in P \setminus \{0\}$.
(2) the series $\sum_{n\geq 0} \varphi^n(t)$ converge for all $t \in P \setminus \{0\}$.

From (1), we have $\varphi(0) = 0$, and from (2), we have $\lim_{n\to 0} \varphi^n(t) = 0$ for all $t \in P \setminus \{0\}$.

Lemma 2.14. Let (X, d) be a cone metric space with cone P not necessary to be normal. Then for $a, c, u, v, w \in E$, we have:

(1) If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$.
(2) If $0 \leq u \ll c$ for each $0 \ll c$, then $u = 0$.
(3) If $u \leq v$ and $v \ll w$, then $u \ll w$.

Now, we give the main result of our work which is a generalization of [3, 4] by omitting the assumption of normality condition in their results.

3. Main Results

Theorem 3.1. Let (X, d) be a complete cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \varphi d(x, y), \tag{2}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Sx_n, \text{ for all } n = 0, 1, 2, \ldots.$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from (2), it follows that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Sx_{n-1}, Sx_n) \\ &\leq \varphi\Big(d(x_{n-1}, x_n)\Big) = d(Sx_{n-2}, Sx_{n-1}) \\ &\leq \varphi^2\Big(d(x_{n-2}, x_{n-1})\Big) \\ &\vdots \\ &\leq \varphi^n\Big(d(x_0, x_1)\Big). \end{aligned} \tag{3}$$

It again follows that

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Sx_{n-1}, Sx_{n+1}) \\ &\leq \varphi\Big(d(x_{n-1}, x_{n+1})\Big) = \Big(d(Sx_{n-2}, Sx_n)\Big) \\ &\leq \varphi^2\Big(d(x_{n-2}, x_n)\Big) \\ &\vdots \\ &\leq \varphi^n\Big(d(x_0, x_2)\Big). \end{aligned} \tag{4}$$

It further follows that

$$\begin{aligned} d(x_n, x_{n+3}) &= d(Sx_{n-1}, Sx_{n+2}) \\ &\leq \varphi\Big(d(x_{n-1}, x_{n+2})\Big) = \Big(d(Sx_{n-2}, Sx_{n+1})\Big) \\ &\leq \varphi^2\Big(d(x_{n-2}, x_{n+1})\Big) \\ &\vdots \\ &\leq \varphi^n\Big(d(x_0, x_3)\Big). \end{aligned} \tag{5}$$

Similarly, for $k = 1, 2, 3, \dots,$ we get

$$d(x_n, x_{n+3k+1}) = \varphi^n\Big(d(x_0, x_{3k+1})\Big), \tag{6}$$

$$d(x_n, x_{n+3k+2}) = \varphi^n\Big(d(x_0, x_{3k+2})\Big), \tag{7}$$

$$d(x_n, x_{n+3k+3}) = \varphi^n\Big(d(x_0, x_{3k+3})\Big). \tag{8}$$

By using (3) and pentagonal property, we have

$$\begin{aligned} d(x_0, x_4) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &\leq d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_1)\Big) \\ &\leq \sum_{i=0}^{3} \varphi^i\Big(d(x_0, x_1)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_7) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &\quad + d(x_4, x_5) + d(x_5, x_6) + d(x_6, x_7) \\ &\leq d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_1)\Big) \\ &\quad + \varphi^4\Big(d(x_0, x_1)\Big) + \varphi^5\Big(d(x_0, x_1)\Big) + \varphi^6\Big(d(x_0, x_1)\Big) \\ &\leq \sum_{i=0}^{6} \varphi^i\Big(d(x_0, x_1)\Big). \end{aligned}$$

Now by induction, we obtain for each $k = 1, 2, 3, \dots$

$$d(x_0, x_{3k+1}) \leq \sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1)\Big). \tag{9}$$

Also, by using (3), (4), and pentagonal property, we have

$$\begin{aligned} d(x_0, x_5) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_5) \\ &\leq d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_2)\Big) \\ &\leq \sum_{i=0}^{2} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_2)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_8) &\le d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &\quad + d(x_4, x_5) + d(x_5, x_6) + d(x_6, x_8) \\ &\le d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_1)\Big) \\ &\quad + \varphi^4\Big(d(x_0, x_1)\Big) + \varphi^5\Big(d(x_0, x_1)\Big) + \varphi^6\Big(d(x_0, x_2)\Big) \\ &\le \sum_{i=0}^{5} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^6\Big(d(x_0, x_2)\Big). \end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(x_0, x_{3k+2}) \le \sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^{3k}\Big(d(x_0, x_2)\Big). \tag{10}$$

Again, by using (3), (5), and pentagonal property, we have

$$\begin{aligned} d(x_0, x_6) &\le d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_6) \\ &\le d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_3)\Big) \\ &\le \sum_{i=0}^{2} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_3)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_9) &\le d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &\quad + d(x_4, x_5) + d(x_5, x_6) + d(x_6, x_9) \\ &\le d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_1)\Big) \\ &\quad + \varphi^4\Big(d(x_0, x_1)\Big) + \varphi^5\Big(d(x_0, x_1)\Big) + \varphi^6\Big(d(x_0, x_3)\Big) \\ &\le \sum_{i=0}^{5} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^6\Big(d(x_0, x_3)\Big). \end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(x_0, x_{3k+3}) \le \sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^{3k}\Big(d(x_0, x_3)\Big). \tag{11}$$

Using inequality (6) and (9) for $k = 1, 2, 3, \ldots,$ we have

$$\begin{aligned} d(x_n, x_{n+3k+1}) &\le \varphi^n\Big(d(x_0, x_{3k+1})\Big) \\ &\le \varphi^n \sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1)\Big) \\ &\le \varphi^n\Big[\sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\le \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \end{aligned} \tag{12}$$

Similarly for $k = 1, 2, 3, \ldots$, inequalities (7) and (10) implies that

$$\begin{aligned} d(x_n, x_{n+3k+2}) &\leq \varphi^n\Big(d(x_0, x_{3k+2})\Big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^{3k}\Big(d(x_0, x_2)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big) \\ &\qquad + \varphi^{3k}\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \end{aligned} \tag{13}$$

Again for $k = 1, 2, 3, \ldots$, inequalities (8) and (11) implies that

$$\begin{aligned} d(x_n, x_{n+3k+3}) &\leq \varphi^n\Big(d(x_0, x_{3k+3})\Big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^{3k}\Big(d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big) \\ &\qquad + \varphi^{3k}\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \end{aligned} \tag{14}$$

Thus, by inequality (12), (13), and (14) we have, for each m,

$$d(x_n, x_{n+m}) \leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \tag{15}$$

Since $\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1)+d(x_0, x_2)+d(x_0, x_3)\Big)$ converges (by definition 2.13), where $d(x_0, x_1)+d(x_0, x_2)+d(x_0, x_3) \in P \setminus \{0\}$, and P is closed, then $\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big) \in P \setminus \{0\}$. Hence

$$\lim_{n\to\infty} \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] = 0.$$

Then, for given $c \gg 0$, there is a natural number N_1 such that

$$\varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \ll c, \quad \forall n \geq N_1. \tag{16}$$

Thus, from (15) and (16), we have

$$d(x_n, x_{n+m}) \ll c, \text{ for all } n \geq N_1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_{n-1} = z$ as $n \to \infty$.

We show that $Sz = z$. Given $c \gg 0$, we choose a natural numbers N_2, N_3, N_4 such that $d(z, x_n) \ll \frac{c}{4}, \ \forall n \geq N_2$, $d(x_{n+1}, x_n) \ll \frac{c}{4}, \ \forall n \geq N_3$, and $d(x_{n-1}, z) \ll \frac{c}{4}, \ \forall n \geq N_4$. Since $x_n \neq x_m$ for $n \neq m$, therefore by pentagonal property, we have

$$\begin{aligned} d(Sz, z) &\leq d(Sz, Sx_n) + d(Sx_n, Sx_{n-1}) + d(Sx_{n-1}, Sx_{n-2}) + d(Sx_{n-2}, z) \\ &\leq \varphi\Big(d(z, x_n) + d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, z) \\ &< d(z, x_n) + d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, z). \end{aligned} \tag{17}$$

Hence, from (17),

$$d(Sz, z) \ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \text{ for all } n \geq N,$$

where $N := \max\{N_2, N_3, N_4\}$. Since c is arbitrary we have $d(Sz, z) \ll \frac{c}{m}, \ \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(Sz, z) \to -d(Sz, z)$ as $m \to \infty$. Since P is closed, $-d(Sz, z) \in P$. Hence $d(Sz, z) \in P \cap -P$. by definition of cone we get that $d(Sz, z) = 0$, and so $Sz = z$. Therefore, S has a fixed point that is z in X.

Next we show that z is unique. For suppose z' be another fixed point of S such that $Sz' = z'$. Therefore,

$$d(z, z') = d(Sz, Sz') \leq \varphi\Big(d(z, z')\Big) < d(z, z').$$

Hence $z = z'$. This completes the proof of the theorem.

Corollary 3.2. Let (X, d) be a complete cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(S^m x, S^m y) \leq \varphi d(x, y), \tag{18}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. From Theorem 3.1, we conclude that S^m has a fixed point say z. Hence

$$Sz = S(S^m z) = S^{m+1} z = S^m(Sz). \tag{19}$$

Then Sz is also a fixed point to S^m. By uniqueness of z, we have $Sz = z$.

Corollary 3.3. (see [4])Let (X, d) be a complete cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \lambda d(x, y), \tag{20}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition 2.13. Hence the results follows from Theorem 3.1.

Corollary 3.4. (see [7])Let (X, d) be a complete cone rectangular metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \varphi d(x, y), \tag{21}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.9 and Theorem 3.1.

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