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Strong Convergence for the Split Feasibility Problem in Real Hilbert Space

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Abstract

In this paper, we study the cyclic algorithm for the split common fixed point problem (SCFPP) and multiple set split feasibility problem (MSSFP). Furthermore we proved the strong convergence for the (SCFPP) and (MSSFP) which extend and improve the result of F. Wang and H.K. Xu [9] from a weak convergence to a strong convergence.
Keywords: Convex Feasibility, Split Feasibility, Split Common Fixed Point, Nonexpansive Mapping, Class – τ operator, Iterative Algorithm and Strong Convergence.

1. Introduction

Let H and K be real Hilbert spaces, A: H → K be a bounded linear operator and A^* be an adjoint of A. Given integer's p, r ≥ 1 and also given sequence of nonempty, closed, convex subsets $\{C_i\}_i^p$ *and* $\{Q_j\}_j^r$ of H and K respectively. The convex feasibility problem (CFP) is formulated as finding a point $x^* \epsilon\, H$ satisfying the property:

$$x^* \in \bigcap_i^p C_i\,. \tag{1.1}$$

Note that, CFP (1.1) has received a lot of attention due to its extensive applications in many applied disciplines diverse as approximation theorem, image recovery, signal processing, control theory, biomedical engineering, communication and geophysics (see [1-3] and the reference therein).
The multiple set split feasibility problem (MSSFP) was recently introduce and studied by Censo, Elfving, Kopf and Bortfeld, see [4] and is formulated as finding a point $x^* \epsilon$ H with the property:

$$x^* \in \bigcap_i^p C_i \; and \;\; Ax^* \in \bigcap_j^r Q_j \tag{1.2}$$

If in a MSSFP (1.2) p = r = 1, we get what is called the split feasibility problem (SFP) see [5], which is formulated as finding a point, $x^* \epsilon\, H$ with the property:

$$x^* \in C \; and \;\; Ax^* \in Q \tag{1.3}$$

where C and Q are nonempty, closed and convex subsets of H and K respectively.
Note that, SFP (1.3) and MSSFP (1.2) model image retrieval (see [5]) and intensity - modulated radiation therapy (see [15, 16]) and have recently been studied by many researchers [6, 7 and 17-25] and references therein.
The MSSFP (1.2) can be viewed as a special case of the CFP (1.1) since (1.2) can be rewriting as

$$x^* \in \bigcap_i^{p+r} C_i\,, \qquad C_{p+j} = \{x^* \in H : x^* \in A^{-1}(Q_j),\;\; 1 \le j \le r\}.$$

However, the methodologies for studying the MSSFP (1.2) are actually different from those for the CFP (1.1) in order to avoid usage of the inverse of A. In other word, the method for solving CFP (1.1) may not apply to solve MSSFP (1.2) straight forwardly without involving the inverse of A. The CQ algorithm of Byne [6, 7] is such an example where only the operator of A is used without involving the inverse.
Since every closed convex subset of Hilbert space is the fixed point set of its associating projection, the CFP (1.1) becomes a special case of the common fixed point problem (CFPP) of finding a point $x^* \epsilon$ H with property:

$$x^* \in \bigcap_i^p Fix(T_i). \tag{1.4}$$

where each $T_i : H \to H$ are some (nonlinear) mapping. Similarly the MSSFP (1.2) becomes a special case of the split common fixed point problem (SCFPP) [8] of finding a point $x^* \epsilon$ H with the property:

$$x^* \in \bigcap_i^p Fix(U_i)\ and\ Ax^* \in \bigcap_j^r Fix(T_j) \quad (1.5)$$

where each, $U_i: H_1 \to H_1$ $(i = 1, 2, 3 \dots p)$ and $T_j: H_2 \to H_2$ $(j = 1, 2, 3 \dots r)$ are some nonlinear operators. If p = r = 1, problem (1.5) is reduces to find a point $x^* \epsilon H$ with property:

$x^* \in Fix(U)\ and\ Ax^* \in Fix(T)$ (1.6)

This is usually called the two-set SCFPP.

The concept of SCFPP in finite dimensional Hilbert space was first introduce by Censor and Segal (see [8]) who invented an algorithm of the two-set SCFPP which generate a sequence $\{x_n\}$ according to the following iterative procedure:

$x_{n+1} = U(x_n + \gamma A^*(T - I)Ax_n), n \geq 0,$ (1.7)

where the initial guess $x_0\ \epsilon\ H$ is choosing arbitrarily and $0 < \gamma < \frac{1}{\|A\|^2}$. By making used of product pace technique, Censor and Segal [8] introduced another algorithm for the general SCFPP (1.5) which generate a Sequence $\{x_n\}$ through the following parallel iterative algorithm:

$x_{n+1} = x_n + \gamma\left(\sum_{i=1}(U_i - I)x_n + \sum_{j=1} \beta_j\left(T_j - I\right)Ax_n\right)$ (1.8)

where $0 < \gamma < 2/L$ with $L = \sum_{i=1}^{p} \alpha_i + \left(\sum_{j=1}^{r} \beta_j\right)\|A\|^2$.

Under suitable assumption impose on parameters $\{\alpha_i\}$ and $\{\beta_j\}$ and for a particular class of operators (called directed operators, see section 2), Censor and Segal [8] proved convergence of algorithm (1.7) and (1.8) to solution of problem (1.6) and (1.5) respectively, in a finite dimensional Hilbert space.

Evidently, problem (1.6) is a particular case of the general SCFPP (1.5). However the corresponding algorithm (1.8) for the general SCFPP (1.5) does not reduce to algorithm (1.7) for problem (1.6).

It was in 2011, F. Wang and H.K. Xu [9] that introduced a new algorithm for solving problem (1.5) which included algorithm (1.7) as a special case for two-set SCFPP (1.6) and convert the SCFPP (1.5) to an equivalent common fixed point problem.

More precisely, they introduced for each $1 \leq j \leq r$, a mapping V_j define as

$$V_j = I + (1/\|A\|)A^*(T_i - I)A$$

and showed that SCFPP (1.5) is equivalent to the common fixed point problem:

$x^* \in Fix(V_i)\ and\ Ax^* \in Fix(T_j)$ (1.9)

This conversion enables one to solve SCFPP (1.5) by applying the existing iterative algorithm for solving the common fixed point problem (1.9).

Motivated by these results, in this paper we extend and improved the result of F. Wang and H.K. Xu [9] from a weak convergence to a strong convergence

2. Preliminaries

Throughout this paper, we adopt the notation:

- I: the identity operator on Hilbert space H.
- Fix (T): the set of fixed point of an operator $T: H \to H$
- Ω: The solution set of SCFPP (1.5).
- $\omega_\omega(x_n)$: The set of the cluster point of x_n in the weak topology i.e. $\left\{\exists\ x_{n_j} \text{ of } x_n \text{ such that } x_{n_j} \rightharpoonup x\right\}$
- $x_n \to x$: $\{x_n\}$ Converge in norm to x
- $x_n \rightharpoonup x$: $\{x_n\}$ Converge weakly to x

Definition 2.1 Assume that C is a closed convex nonempty subset of a real Hilbert space H. A sequence $\{x_n\}$ in H is said to be Fejer monotone with respect to C if and only if $\|x_{n+1} - z\| \leq \|x_n - z\|,\ for\ all\ n \geq 1\ and\ z \in C$

Definition 2.2 let $T: H \to H$ be an operator. We say that $(I - T)$ is demiclosed at zero, if for any sequence $\{x_n\}$ in H, there holds the following implication:

$x_n \rightharpoonup x$ and $(I - T)x_n \to 0\ as\ n \to \infty$, then $(I - T)x = 0$.

Definition 2.3 Let H be a real Hilbert space and let $x, y\ \epsilon\ H$. Following Haugazeau [10], we use $H(x, y)$ to denote the half - space determined by x, y; namely, $H(x, y) = \{\langle u - y, x - y\rangle \leq 0\}$

Definition 2.4 An operator T : H → H is said to be a class – τ operator, if for each $x\ \epsilon\ H, Fix(T) \subseteq H(x, T(x))\ or\ equivalently, \langle z - Tx, x - Tx\rangle \leq 0, for\ all\ z\ \epsilon\ Fix\ (T)\ and\ x\ \epsilon\ H.$

Remark 2.5 A class-τ operator is also called directed operator see [8, 12], separating operator see [13] or cutter operator see [14]. Class-τ operators are important because they include many type of nonlinear operators arising in applied mathematics such as approximation theorem and convex optimization theorem.

Definition 2.6 An operator $T: H \to H$ is said to be

(a) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, for\ all\ x, y \in H$

(b) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\|$, for all $x \in H\ and\ z \in Fix(T)$

(c) strictly quasi - nonexpansive if Fix(T)$\neq \emptyset$ and $\|Tx - z\| < \|x - z\|$, for all $x \in H/Fix(T)\ and\ z \in Fix(T)$

(d) α -strongly quasi-nonexpansive if there exist $\alpha > 0$ with the property: $\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.

The operator $T_\lambda = (1 - \lambda)\ I + \lambda T$, $\lambda \epsilon (0, 2)$ is called a relaxation of T.

Lemma 2.7 [26] Let $\{x_n\}$ be a sequence in a Banach space E. We have the following result:

(i)) $x_n \rightharpoonup x$, $\Leftrightarrow$ $f(x_n) \to f(x)$ for each $f \epsilon E^*$;

(ii) $x_n \to$ x $\Rightarrow x_n \rightharpoonup x$;

(iii) $x_n \rightharpoonup x$, $\Rightarrow \{x_n\}$ is bounded and $\|x\| \leq$ limimf$_{n\to\infty}\|x_n\|$

(iv) $x_n \rightharpoonup x$ (In E), $f_n \to f$ (in E^*),$\Rightarrow f(x_n) \to f(x)\ (in\ R)$.

Remark 2.8: Lemma 2.7 (ii) Show that strong convergence implies weak convergence. The converse however is false i.e. weak convergence does not imply strong convergence.

Lemma 2.9 [26] Let E be a finite dimensional normed linear space, then the weak and strong topologies coincide. (In particular, a sequence $\{x_n\}$ in E converges weakly if and only if it converges strongly).

Lemma 2.10 [9] let $T : H \to H$ be an operator. Then the following statements are equivalent

(i) T is class - τ operator;

(ii) $\|x - Tx\| \leq \langle x - z, x - Tx\rangle, z \epsilon Fix(T)\ and\ x \epsilon H$;

(iii) There hold the relation : $\|z - Tx\|^2 \leq \|z - x\|^2 - \|x - Tx\|^2$, $z \epsilon Fix(T)\ and\ x \epsilon H$.

Consequently a class - τ operator is 1 - strongly quasi - nonexpansive.

Lemma 2.10 [14, 11]

(i) The fixed point set of a class - τ operator T, is closed and convex, indeed

$$Fix(T) = \bigcap_{x\epsilon H} H(x, Tx)\,.$$

(ii) If T is class – τ operator, then so is the relaxation of $T_\lambda\ for\ \lambda \epsilon (0,1)$.

(iii) T is class - τ operator if and only if its relaxation of T, furthermore T_λ is $\frac{2-\lambda}{\lambda}$ strongly - nonexpansive, i.e. $\|z - Tx\|^2 \leq \|z - x\|^2 - \frac{2-\lambda}{\lambda}\|x - Tx\|^2$, $z \epsilon Fix(T)\ and\ x \epsilon H$.

Lemma 2.11 [3] If a sequence $\{x_n\}$ is fejer monotone with respect to a closed convex nonempty subset C, then the following hold.

(i) $x_n \rightharpoonup x \in C$ if and only if $\omega_\omega \subset C$;

(ii) The sequence $\{P_\Omega x_n\}$ converges strongly to some point in C;

(iii) If $x_n \rightharpoonup x \in C$, then $x = \lim_{n\to\infty} P_c x_n$.

Lemma 2.12 [9] Let $A : H \to K$ be a given bounded linear operator and $T : K \to K$ be a class –τ operator on K. Assume that the equation $(I - T)Ax = 0$ (a)

has a nonempty solution set, then for each constant $0 < \sigma \leq \frac{1}{\|A\|^2}$, the operator:

$V := I + \sigma A^*(T - I)A$ (b)

is class – τ operator on H; moreover

$Fix(V) = \{x \epsilon H : Ax \epsilon Fix(T)\} = A^{-1}(Fix(T))$. (c)

3. Main Results

Theorem 3.1 Let $U_i\ and\ V_i$ be class - τ operators on real Hilbert space H for (i = 1, 2, 3,…, p), suppose that $U_i - I$ and $V_i - I$ are demiclosed at zero for every i = 1, 2, 3,…, p. Assume that the solution set Ω of problem (1.9) (with r = p) is nonempty and let P_Ω be a metric projection of H onto Ω satisfying $\langle x_n - x^*, x_n - P_\Omega x_n\rangle \leq 0$. Then the sequence $\{x_n\}$ define by

x_{n+1}= $U_{[n]}$ [x_n+ λ($V_{[n]}$ x_n- x_n)]

converges strongly to a point $x^* \epsilon \Omega$, where [n]:= n (mod p) with mod function taking value in the set $\{1, 2, 3 \ldots p\}, \lambda \epsilon (0;\ 1)\ and\ x_0 \epsilon H$ is choosing arbitrarily.

Proof. To show that $x_n \to x^*$, it suffices to show that $x_n \rightharpoonup x^*$ and $\|x_n\| \to \|x^*\|$ as $n \to \infty$.

As we are in Hilbert space, now, taking $x^* \in \Omega$ and let $V_{\lambda,n} = I + (V_{[n]} - I)$, since $V_{\lambda,n}$ is $\frac{2-\lambda}{2}$ strongly quasi-nonexpansive, we deduce from lemma 2.10 (iii) that

$$\|x_{n+1} - x^*\|^2 = \left\|U_{[n]}[x_n + \lambda(V_{[n]} - I)x_n] - x^*\right\|^2$$

$$=\left\|U_{[n]}V_{\lambda,n}x_n - x^*\right\|^2$$
$$\le \left\|V_{\lambda,n}x_n - x^*\right\|^2 - \left\|U_{[n]}V_{\lambda,n} - V_{\lambda,n}\right\|^2$$
$$\le \left\|V_{\lambda,n}x_n - x^*\right\|^2$$
$$\le \|x_n - x^*\|^2 - \frac{2-\lambda}{2}\left\|V_{\lambda,n}x_n - x_n\right\|^2$$
$$= \|x_n - x^*\|^2 - \frac{2-\lambda}{2}\left\|\left(I + \left(V_{[n]} - I\right)\right)x_n - x_n\right\|^2$$
$$= \|x_n - x^*\|^2 - \lambda\,(2-\lambda)\left\|V_{[n]}x_n - x_n\right\|^2.$$

Thus $\{x_n\}$ is a Fejer monotone with respect to Ω and

$$\sum_{n\ge 1}\left\|V_{[n]}x_n - x_n\right\|^2 < \infty.$$

In particular, we have

$$\left\|V_{[n]}x_n - x_n\right\| \to 0$$

It also follow from lemma 2.10 (iii) that

$$\|x_{n+1} - x_n\|^2 = \left\|U_{[n]}V_{\lambda,n}x_n - V_{\lambda,n}x_n + V_{\lambda,n}x_n - x_n\right\|^2$$
$$\le \left(\left\|U_{[n]}V_{\lambda,n}x_n - V_{\lambda,n}x_n\right\| + \left\|V_{\lambda,n}x_n - x_n\right\|\right)^2$$
$$= \left\|U_{[n]}V_{\lambda,n}x_n - V_{\lambda,n}x_n\right\|^2 + 2\left\|V_{\lambda,n}x_n - x_n\right\|\left\|U_{[n]}V_{\lambda,n}x_n - V_{\lambda,n}x_n\right\| + \left\|V_{\lambda,n}x_n - x_n\right\|^2$$
$$\le 2\left(\left\|U_{[n]}V_{\lambda,n}x_n - V_{\lambda,n}x_n\right\|^2 + \left\|V_{\lambda,n}x_n - x_n\right\|^2\right)$$
$$\le 2\left(\left\|V_{\lambda,n}x_n - x^*\right\|^2 - \left\|U_{[n]}V_{\lambda,n}x_n - x^*\right\|^2 + \|x_n - x^*\|^2 - \left\|V_{\lambda,n}x_n - x^*\right\|^2\right)$$
$$= 2\left(\|x_n - x^*\|^2 - \left\|U_{[n]}V_{\lambda,n}x_n - x^*\right\|^2\right)$$
$$\le 2(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2)$$

$$\Rightarrow \|x_{n+1} - x_n\|^2 \le 2(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2)$$
$$\Rightarrow \sum_{n>0}\|x_{n+1} - x_n\|^2 < \infty.$$

Now, let $x^* \in \omega_\omega(x_n)$ and let an index $i \in \{1, 2, 3, \dots, n\}$ be fixed, noticing that the pool indexes is finite, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$, and $[n_k] = i$ for all k.
It turns out that
$\left\|V_i x_{n_k} - x_{n_k}\right\| = \left\|V_{[n_k]}x_{n_k} - x_{n_k}\right\| \to 0$, as k → ∞.
By demiclosedness of $(V_i - I)$ at zero, we get $x^* \in Fix(V_i)$.
Set $y_{n_k} = x_{n_k} + \lambda(V_{[n]} - I)x_{n_k}$, then $y_{n_k} \rightharpoonup x^*$, as $k \to \infty$, since $\left\|V_i x_{n_k} - x_{n_k}\right\| \to 0$, as k → ∞,
it follows that

$$\left\|x_{n_{k+1}} - x^*\right\|^2 = \left\|U_i\left[x_{n_k} + \lambda\left(V_{[n]} - I\right)x_{n_k}\right] - x^*\right\|^2$$
$$= \left\|U_i y_{n_k} - x^*\right\|^2$$
$$\le \left\|y_{n_k} - x^*\right\|^2 - \left\|U_i y_{n_k} - y_{n_k}\right\|^2$$
$$\le \left\|y_{n_k} - x^*\right\|^2 = \left\|V_{\lambda,i}x_{n_k} - x^*\right\|^2$$
$$\le \left\|x_{n_k} - x^*\right\|^2 - \frac{2-\lambda}{2}\left\|V_{\lambda,i}x_{n_k} - x_{n_k}\right\|^2$$
$$= \left\|x_{n_k} - x^*\right\|^2 - \frac{2-\lambda}{2}\left\|\left(I + \left(V_{[n]} - I\right)\right)x_{n_k} - x_{n_k}\right\|^2$$
$$= \left\|x_{n_k} - x^*\right\|^2 - \lambda\,(2-\lambda)\left\|V_{[n]}x_{n_k} - x_{n_k}\right\|^2$$

$$\Rightarrow \left\|x_{n_{k+1}} - x^*\right\|^2 \le \left\|y_{n_k} - x^*\right\|^2 \le \left\|x_{n_k} - x^*\right\|^2.$$

Hence $\lim_{k\to\infty}\left\|x_{n_k} - x^*\right\|^2$ *coinside with* $\lim_{k\to\infty}\left\|y_{n_k} - x^*\right\|^2$, moreover
$\left\|U_i y_{n_k} - y_{n_k}\right\|^2 \le \left\|y_{n_k} - x^*\right\|^2 - \left\|U_i y_{n_k} - x^*\right\|^2 = \left\|y_{n_k} - x^*\right\|^2 - \left\|x_{n_{k+1}} - x^*\right\|^2 \to 0$, as k→ ∞.
Since $(U_i - I)$ is demiclosed at zero, *we have* $x^* \in Fix(U_i)$. Since this is true for every i, we get that $\omega_\omega(x_n) \subset \Omega$. By lemma 2.11 we conclude that the sequence (x_n) converges weakly to a point $x^* \in \Omega$ i.e.
$x_n \rightharpoonup x^*$ as $n \to \infty$. (3.1.1)
Next, we show that
$\|x_n\| \to \|x^*\|$ as $n \to \infty$,
to show this,

it suffices to show that
$\|x_{n+1}\| \to \|x^*\|$ as n → ∞.
Now, since $\{x_n\}$ is fejer monotone, we deduce that

$$|\|x_{n+1}\| - \|x^*\||^2 \le \|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2.$$

Therefore, we have
$|\|x_{n+1}\| - \|x^*\||^2 \le \|x_n - x^*\|^2$
$\Longrightarrow |\|x_{n+1}\| - \|x^*\|| \le \|x_n - x^*\| = \|x_n - P_\Omega x_n + P_\Omega x_n - x^*\|$
$\le \|x_n - P_\Omega x_n\| + \|P_\Omega x_n - x^*\|$ (3.1.2)
Claim $\|x_n - P_\Omega x_n\| \le \|P_\Omega x_n - x^*\|$
Proof of claim

$$\|x_n - P_\Omega x_n\|^2 = \|x_n - x^* + x^* - P_\Omega x_n\|^2$$
$$= \|x_n - x^*\|^2 + 2\langle x_n - x^*, x^* - P_\Omega x_n\rangle + \|x^* - P_\Omega x_n\|^2$$
$$= \|x_n - x^*\|^2 + 2\langle x_n - x^*, x^* - x_n + x_n - P_\Omega x_n\rangle + \|x^* - P_\Omega x_n\|^2$$
$$= \|x_n - x^*\|^2 + 2\langle x_n - x^*, x^* - x_n\rangle + 2\langle x_n - x^*, x_n - P_\Omega x_n\rangle + \|x^* - P_\Omega x_n\|^2$$
$$= -\|x_n - x^*\|^2 + 2\langle x_n - x^*, x_n - P_\Omega x_n\rangle + \|x^* - P_\Omega x_n\|^2$$
$$\le \|x^* - P_\Omega x_n\|^2$$

$\Longrightarrow \|x_n - P_\Omega x_n\|^2 \le \|x^* - P_\Omega x_n\|^2$ (3.1.3)
Now, put (3.1.3) in (3.1.2), it follows that
$|\|x_{n+1}\| - \|x^*\|| \le 2\|x^* - P_\Omega x_n\|$
$\Longrightarrow 0 \le \limsup_{n\to\infty} |\|x_{n+1}\| - \|x^*\|| \ \limsup_{n\to\infty} 2\|x^* - P_\Omega x_n\| = 0$
$\Longrightarrow \limsup_{k\to\infty} |\|x_{n+1}\| - \|x^*\|| = 0.$
Hence
$\|x_{n+1}\| \to \|x^*\|$, as n → ∞. (3.1.4)
By (3.1.1) and (3.1.4), we have that
$x_n \to x^*$ as n → ∞. ■

Theorem 3.2 Let P ≥ 1 and be an integer and let $\{U_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^p$, be a family of class-τ operators on a real Hilbert space H and K respectively. Suppose that SCFPP (1.5) with (r = p) has a nonempty solution set Ω and let P_Ω be a metric projection of H onto Ω satisfying $\langle x_n - x^*, x_n - P_\Omega x_n\rangle \le 0$, suppose also for each $1 \le i \le p$, $(U_i - I)$ *and* $(T_i - I)$ are both demiclosed. Then the sequence $\{x_n\}$ define by
$x_{n+1} = U_{[n]}[x_n + \gamma A^*(T_{[n]} - I)x_n]$ (3.2.1)
converges strongly to a point $x^* \in \Omega$, where [n]:= n (mod p) with mod function taking value in the set $\{1, 2, 3, \dots, p\}$, $0 < \gamma \le \frac{1}{\|A\|^2}$ and $x_0 \epsilon H$ is choosing arbitrarily.

Proof. Take $0 < \sigma \le \frac{1}{\|A\|^2}$ such that $\frac{\gamma}{\sigma} < 1$ e.g. ($\sigma = \frac{1}{\|A\|^2}$), set $V_{\lambda,i} = I + \sigma A^*(T_i - I)A$, for $i = 1, 2, 3, \dots, p$ *and* $[n] = i$.
By lemma 2.12, V_i is class - τ operator. Let $U_{[n]} = U_{n(mod\ p)}$ and $V_{[n]} = V_{n(mod\ p)}$.
We can rewrite (3.2.1) as
$x_{n+1} = U_{[n]}[x_n + \lambda(V_{[n]} - I)x_n]$
where $\lambda = \frac{\gamma}{\sigma} \in (0,1)$.
We next prove the demiclosedness (at zero) of the operator $(V_i - I)$ for every $i = 1, 2, 3, \dots, p$.
To see this, assume that $z_n \rightharpoonup z$ and $(I - V_I)z_n \to 0$ as n → ∞.

$$\|z_n - V_I z_n\| = \|z_n - (I + \sigma A^*(T_i - I)A)z_n\|$$
$$= \|\sigma A^*(T_i - I)Az_n\|$$
$$= \sigma\|A^*(T_i - I)Az_n\|$$

$\Longrightarrow \|A^*(T_i - I)Az_n\| = \frac{1}{\sigma}\|(I - V_I)z_n\| \to 0.$ (3. 2. 2)
Now take q∈ Ω. Since T_i is class - τ operator, we arrive at

$$\|(T_i - I)Az_n\|^2 = \langle (T_i - I)Az_n, (T_i - I)Az_n\rangle$$
$$= \langle T_i Az_n - Aq + Aq - Az_n, (T_i - I)Az_n\rangle$$
$$= \langle T_i Az_n - Aq, (T_i - I)Az_n\rangle + \langle Aq - Az_n, (T_i - I)Az_n\rangle$$
$$= \langle T_i Az_n - Aq, (T_i - I)Az_n\rangle + \langle q - z_n, A^*(T_i - I)Az_n\rangle$$
$$\le \langle q - z_n, A^*(T_i - I)Az_n\rangle$$
$$\le \|q - z_n\|\|A^*(T_i - I)Az_n\|$$
$$\le M\|A^*(T_i - I)Az_n\|$$

where M is constant such that $\|q - z_n\| \le M$ for all n. It turns out from (3.2.2) that:

$\|(T_i - I)Az_n\| \to 0, as\ n \to \infty.$

However, the weak continuity of A yield that $Az_n \rightharpoonup Az$, which together with the demiclosedness of $(T_i - I)$ at zero enables us to deduce that

$Az = T_i Az \Longrightarrow Az \in Fix(T_i)$
$\Longrightarrow z \in A^{-1}(Fix(T_i))$
$\Longrightarrow z \in Fix(V_i).$

This show that $(V_i - I)$ is demiclosed at zero for every $i = 1, 2, 3, \dots, p$.

Being generated by algorithm (3.2.1), the sequence $\{x_n\}$ is seen to converge strongly to a point $x^* \in \Omega$, by virtue of Lemma 2.12 and Theorem 3.1.

Corollary 3.3

Let P ≥ 1 and be an integer and let $\{U_i\}_{i=1}^{p}$ and $\{T_j\}_{j=1}^{p}$, be a family of class-τ operators on a real Hilbert space H and K respectively. Suppose that MSSFP (1.2) with (r = p) has a nonempty solution set Ω and let P_Ω be a metric projection of H onto Ω satisfying $\langle x_n - x^*, x_n - P_\Omega x_n \rangle \le 0$, suppose also for each $1 \le i \le p$, $(U_i - I)$ and $(T_i - I)$ are both demiclosed. Then the sequence $\{x_n\}$ define by

$$x_{n+1} = P_{c_{[n]}}\left[x_n + \gamma A^*\left(P_{Q_{[n]}} - I\right)x_n\right] \qquad (3.3.1)$$

Converges strongly to a point $x^* \in \Omega$. Where [n]:= n (mod p) with mod function taking value in the set $\{1, 2, 3, \dots, p\}$, $0 < \gamma \le \frac{1}{\|A\|^2}$ and $x_0 \in H$ is choosing arbitrarily.

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