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Article

A New General Iterative Method for an Infinite Family of Nonexpansive Mappings in Hilbert Spaces

Abba Auwalu

Government Day Secondary School, P.M.B. 1008, Gumel, Ministry Of Education, Science and Technology, Dutse, Jigawa State, Nigeria; E-mail: abbaauwalu@yahoo.com

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Abstract: In this article, by using the W-mapping, η-strongly monotone and L-Lipschitzian operator, we introduce and study a new iterative scheme with Meir-Keeler contraction for finding a common fixed point of an infinite family of nonexpansive mappings in the frame work of Hilbert spaces. We prove the strong convergence of the proposed iterative scheme to the unique solution of some variational inequality. The methods in this article are interesting and different from those given in many other articles. Our results improve and extend the corresponding results announced by many authors.

Keywords: Hilbert space; nonexpansive mapping; W-mapping; η-strongly monotone and L-Lipschitzian operator; variational inequality; Meir-Keeler contraction; fixed point.

Mathematics Subject Classification (2000): 47H05, 47H09, 47J05, 47J25.

1. Introduction

Let C be a closed convex nonempty subset of a real Hilbert space H with the inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively.
Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

A point $x \in C$ is called a *fixed point* of T if and only if $Tx = x$. We denote by $F(T)$ the set of fixed points of the mapping T, that is

$$F(T) = \{x \in C : Tx = x\}. \tag{2}$$

We assume that $F(T) \neq \emptyset$. It is well known that $F(T)$ is closed and convex (see e.g., Goebel and Kirk [3]).

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A self mapping $f : C \to C$ is said to be *contraction* if there is a constant $\alpha \in [0, 1)$ such that

$$\|fx - fy\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{3}$$

We use $\prod_C$ to denote the collection of all contractions on C.
Let $F : C \to C$ be an operator. F is called L-*Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in C. \tag{4}$$

The map F is said to be *monotone* if

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{5}$$

The mapping F is said to be η-*strongly monotone* if there exists $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{6}$$

An operator $A : H \to H$ is said to be strongly positive if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{7}$$

Remark 1.1 *From the definition of* A, *we note that a strongly positive bounded linear operator* A *is a* $\|A\|$-*Lipschitzian and* $\overline{\gamma}$-*strongly monotone operator.*

Let C be a nonempty closed convex subset of a real Hilbert space H, and $F : C \to C$ be a nonlinear map. Then, a *variational inequality problem* with respect to C and F is to find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{8}$$

We denote by $VI(F, C)$ the set of solutions of this variational inequality problem.
The variational inequality problem was initially introduced and studied by Stampacchia [17] in 1964. It is well known that variational inequalities cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, (see [1-23]).
It is also known that the $VI(F, C)$ is equivalent to the fixed point equation

$$x^* = P_C[x^* - \mu F(x^*)], \tag{9}$$

where P_C is the metric (nearest point) projection of H onto C (i.e., $P_C x = y$ where $\|x - y\| = \inf\{\|x - z\| : z \in C\}$ for $x \in H$) and $\mu > 0$ is an arbitrarily fixed constant.
Consequently, under appropriate conditions on F and μ, fixed point methods can be used to find or approximate a solution of the variational inequality.

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Iterative methods for approximating fixed points of nonexpansive mappings and their generalizations which solves some variational inequalities problems have been studied by a number of authors (e.g., see [5, 9, 13, 15, 20, 23] and the references therein).

In 1953, Mann [4] introduced a well-known classical iteration to approximate a fixed point of a nonexpansive mapping. This iteration is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n), \quad n \geq 0, \tag{10}$$

where the initial guess x_0 is taken in C arbitrarily, and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0,1]$. But Mann's iteration process has only weak convergence, even in Hilbert space setting. In general for example, Reich [12] showed that if E is a uniformly convex Banach space and has a Fréchet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the process (10) converges weakly to a point in $F(T)$. Therefore, many authors try to modify mann's iteration process to have strong convergence for nonlinear operators. (e.g., see [5, 9, 13, 15, 16, 20, 21, 23] and the references therein).

In 2005, Kim and Xu [9] introduced the following iteration process:

$$\begin{aligned} x_0 &= x \in C \ \ \textit{arbitrarily chosen}, \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \quad n \geq 0. \end{aligned} \tag{11}$$

They proved in uniformly smooth Banach space that the sequence $\{x_n\}$ defined by (11) converges strongly to a fixed point of T under some appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

In 2006, Marino and Xu [5] introduced the following iterative algorithm:

$$x_0 = x \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{12}$$

where T is a self-nonexpansive mapping on H, f is a contraction and A is a strong positive linear bounded operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (12) converges strongly to the unique solution of the variational inequality:

$$\langle (A-\gamma f)\tilde{x}, \tilde{x} - x\rangle \leq 0, \quad \forall x \in C, \tag{13}$$

which is also the optimality condition for the minimization problem $\underset{x\in C}{min}\frac{1}{2}\langle Ax, x\rangle - h\langle x\rangle$, where h is a potential function for γf $(i.e., h'(x) = \gamma f(x),$ for $x \in H)$.

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In 2008, Yao et al. [23] modified Mann's iterative scheme by using the so-called viscosity approximation method which was introduced by Moudafi [7]. More precisely, Yao et al. [23] introduced and studied the following iterative algorithm:

$$\begin{cases} x_0 = x \in K, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n)y_n, \quad n \geq 0, \end{cases} \tag{14}$$

where T is a nonexpansive mapping of K into itself and f is a contraction on K. They obtained a strong convergence theorem under some mild restrictions on the parameters.

In 2010, Tian [20] considered the following general iterative method:

$$x_0 = x \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \geq 0, \tag{15}$$

where T is a nonexpansive mapping on H, f is a contraction, F is k-Lipschitzian and η-strongly monotone with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$.
He proved that if the sequence $\{\alpha_n\}$ of parameters satisfy some appropriate conditions, then the sequence $\{x_n\}$ generated by (15) converges strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall\, x \in F(T).$$

In 1999, Atsushiba and Takahashi [1] defined the mapping J_n as follows:

$$\begin{aligned} U_{n,1} &= \gamma_{n,1}T_1 + (1-\gamma_{n,1})I, \\ U_{n,2} &= \gamma_{n,2}T_2U_{n,1} + (1-\gamma_{n,2})I \\ U_{n,3} &= \gamma_{n,3}T_3U_{n,2} + (1-\gamma_{n,3})I \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1}T_{N-1}U_{n,N-2} + (1-\gamma_{n,N-1})I \\ J_n = U_{n,N} &= \gamma_{n,N}T_NU_{n,N-1} + (1-\gamma_{n,N})I. \end{aligned} \tag{16}$$

Where $I = U_{n,0}$ and $\{\gamma_{n,i}\}_i^N \subseteq [0,1]$. This mapping is called the *J-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$.

In 2000, Takahashi and Shimoji [19] proved that if X is strictly convex Banach space, then $F(J_n) = \bigcap_{i=1}^{N} F(T_i)$, where $0 < \gamma_{n,i} < 1$.

In 2007, Shang et al. [13] introduced a composite iteration scheme as follows:

$$\begin{aligned} x_0 &= x \in C \;\; arbitrarily \;\; chosen, \\ y_n &= \beta_n x_n + (1-\beta_n)J_nx_n, \\ x_{n+1} &= \alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n, \quad n \geq 0, \end{aligned} \tag{17}$$

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where $f \in \prod_C$ is a contraction, and A is a linear bounded operator.
But, the iterative scheme (17) is not well-defined because x_n $(n \geq 1)$ may not lie in C, so $J_n x_n$ is not defined. However, if $C = H$, the iterative scheme (17) is well-defined and Theorem 2.1 of [13] is obtained. In the case $C \neq H$, the iterative scheme (17) have to be modified in order to make it well-defined.

In 2009, Kangtunyakarn and Suantai [8] introduced a new mapping, called *K-mapping*, for finding a common fixed point of a finite family of nonexpansive mappings. For a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and sequence $\{\gamma_{n,i}\}_{i=1}^N$ in $[0,1]$, the mapping $K_n : C \to C$ defined as follows:

$$\begin{aligned}
U_{n,1} &= \gamma_{n,1}T_1 + (1-\gamma_{n,1})I, \\
U_{n,2} &= \gamma_{n,2}T_2U_{n,1} + (1-\gamma_{n,2})U_{n,1}, \\
U_{n,3} &= \gamma_{n,3}T_3U_{n,2} + (1-\gamma_{n,3})U_{n,2}, \\
&\vdots \\
U_{n,N-1} &= \gamma_{n,N-1}T_{N-1}U_{n,N-2} + (1-\gamma_{n,N-1})U_{n,N-2}, \\
K_n = U_{n,N} &= \gamma_{n,N}T_NU_{n,N-1} + (1-\gamma_{n,N})U_{n,N-1}.
\end{aligned} \tag{18}$$

The mapping K_n is called *K-mapping* generated by $T_1, T_2, T_3, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}, \ldots, \gamma_{n,N}$.

Recently, in 2010, Singthong and Suantai [15] introduced a composite iterative scheme as follows:

$$\begin{aligned}
x_0 &= x \in C \;\; \textit{arbitrarily chosen}, \\
y_n &= \beta_n x_n + (1-\beta_n)K_n x_n, \\
x_{n+1} &= P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n\big), \quad n \geq 0,
\end{aligned} \tag{19}$$

where $f \in \prod_C$ is a contraction, and A is a bounded linear operator. They proved, under certain appropriate conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (19) converges strongly to a common fixed point q of the finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, which solved the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad p \in \bigcap_{i=1}^N F(T_i). \tag{20}$$

Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i\}_{i=1}^\infty$ be a countable infinite family of nonexpansive mappings of C into itself and let $\{\lambda_i\}_{i=1}^\infty$ be a sequence of real numbers such that $\lambda_i \in (0,1)$ for all $i \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of Natural numbers. For all $n \in \mathbb{N}$, define

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a mapping $W_n : C \to C$ by

$$\begin{aligned}
U_{n,n+1} &:= I,\\
U_{n,n} &:= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &:= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &:= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &:= \lambda_{k-1} T_{n-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &:= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n := U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{21}$$

The mapping W_n is called the *W-mapping* generated by the countable infinite family of nonexpansive mappings $T_1, T_2, T_3, \ldots$ and $\lambda_1, \lambda_2, \lambda_3 \ldots$.

The following famous theorem is referred to as the Banach contraction principle.

Theorem 1.2 (Banach, [2]) *Let (X, d) be a complete metric space and let f be a contraction on X, i.e., there exists $r \in (0,1)$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Then f has a unique fixed point.*

Theorem 1.3 (Meir and Keeler, [6]) *Let (X, d) be a complete metric space and let ϕ be a Meir-Keeler contraction (MKC) on X, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \epsilon + \delta$ implies $d(\phi(x), \phi(y)) < \epsilon$ for all $x, y \in X$. Then ϕ has a unique fixed point.*

Remark 1.4 *Theorem 1.3 is one of generalizations of Theorem 1.2, because contractions are MKCs.*

Question 1. Can Theorem 3.4 of Marino and Xu [5], Theorem 1 of Yao et al. [23], Theorem 2.1 of Singthong and Suantai [15], and so on be extended from one or finite family of nonexpansive mappings to countable infinite family of nonexpansive mappings?
Question 2. We know that the Meir-Keeler contraction (MKC) is more general than the contraction. What happens if the contraction is replaced by the MKC?
Question 3. We know that the η-strongly monotone and L-Lipschitzian operator is more general than the strong positive linear bounded operator. What happens if the strong positive linear bounded operator is replaced by the η-strongly monotone and L-Lipschitzian operator?

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Question 4. Can the restriction imposed on the parameter $\{\lambda_{n,i}\}$ in [13, 15] be relaxed?

The purpose of this paper is to give the affirmative answers to these questions mentioned above. Motivated by Kim and Xu [9], Marino and Xu [5], Tian [20], Yao et al. [23], Shang et al. [13], Singthong and Suantai [15], we introduced a general iterative scheme as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = P_C\big[\alpha_n \gamma \phi(x_n) + (I - \alpha_n \mu F)y_n\big], \quad n \geq 0, \end{cases} \tag{22}$$

where P_C is the projection of H onto C, W_n is the *W-mapping* generated by the countable infinite family of nonexpansive mappings $T_1, T_2, T_3, \ldots$ and $\lambda_1, \lambda_2, \lambda_3 \ldots$, ϕ is an MKC and $F : C \to C$ is an η-strongly monotone and L-Lipschitzian operator in Hilbert space. We prove, under certain appropriate conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (22) converges strongly to a common fixed point of the countable infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$, which solves some variational inequality.

2. Preliminaries

In the sequel, we will make use of the following lemmas.

Lemma 2.1 (Xu, [22]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \geq 0,$
where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

(*i*) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (Suzuki, [18]) *Let ϕ be a MKC on a convex subset C of a Banach space E. Then for each $\epsilon > 0$, there exists $r \in (0,1)$ such that $\|x - y\| \geq \epsilon$ implies $\|\phi(x) - \phi(y)\| \leq r\|x - y\|$ for all $x, y \in C$.*

Lemma 2.3 (Demiclosedness Principle, [18]) *Let H be a real Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.4 (Wang, [21]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \to C$ be an η-strongly monotone and L-Lipschitzian operator with $\eta > 0$, $L > 0$. Assume that $0 < \mu < 2\eta/L^2$ and $\tau = \mu(\eta - \frac{\mu L^2}{2})$. Then for each $t \in \big(0, \min\{1, \frac{1}{\tau}\}\big)$, we have*

$$\|(I - t\mu F)x - (I - t\mu F)y\| \leq (1 - t\tau)\|x - y\|, \quad \forall x, y \in C.$$

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Lemma 2.5 (Shimoji and Takahashi, [14]) *Let C be a closed convex nonempty subset of a strictly convex real Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be a countable infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\{\lambda_i\}_{i=1}^{\infty}$ be a real sequence such that $0 < \lambda_i \leq b < 1$ for all $i \in \mathbb{N}$, for some constant $b \in (0,1)$. Let W_n be the* W-mappings *generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Then, W_n is nonexpansive and*

$$\bigcap_{n=1}^{\infty} F(W_n) = \bigcap_{i=1}^{\infty} F(T_i).$$

Lemma 2.6 (Ofoedu, [11]) *Let C be a closed convex nonempty subset of a strictly convex real Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be a countable infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\{\lambda_i\}_{i=1}^{\infty}$ be a real sequence such that $0 < \lambda_i \leq b < 1$ for all $i \in \mathbb{N}$, for some constant $b \in (0,1)$. Suppose $W_n : C \to C$ be given by (21) for all $n \in \mathbb{N}$, then,*

$$\|W_{n+1}x_n - W_n x_n\| \leq 2\prod_{i=1}^{n} \lambda_i \|x_n - q\| \to 0 \text{ as } n \to \infty,$$

for every bounded sequence $\{x_n\} \subset C$, $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

3. Results and Discussion

Lemma 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping suth that $F(T) \neq \emptyset$ and let ϕ be an MKC on C. Suppose $F : C \to C$ be an η-strongly monotone and L-Lipschitzian operator with $\eta > 0$, $L > 0$. Assume that $0 < \mu < 2\eta/L^2$, $t \in \left(0, \min\{1, \frac{1}{\tau}\}\right)$ and $0 < \gamma < \tau$, where $\tau = \mu(\eta - \frac{\mu L^2}{2})$. Then there exists a unique $x_t \in C$ such that*

$$x_t = t\gamma\phi(x_t) + (I - t\mu F)Tx_t.$$

proof *From the definition of MKC, we can see MKC is also a nonexpansive mapping. Now, for each $t \in \left(0, \min\{1, \frac{1}{\tau}\}\right)$, define a mapping T_t on C by*

$$T_t x = t\gamma\phi(x) + (I - t\mu F)Tx, \quad \forall x \in C.$$

Then, by Lemma 2.4, we have

$$\begin{aligned}
\|T_t x - T_t y\| &\leq t\gamma\|\phi(x) - \phi(y)\| + \|(I - t\mu F)Tx - (I - t\mu F)Ty\| \\
&\leq t\gamma\|\phi(x) - \phi(y)\| + (1 - t\tau)\|Tx - Ty\| \\
&\leq t\gamma\|x - y\| + (1 - t\tau)\|x - y\| \\
&= [1 - t(\tau - \gamma)]\|x - y\|,
\end{aligned}$$

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which implies that T_t is a contraction. Hence, T_t has a unique fixed point, denoted by x_t, which uniquely solve the fixed point equation:

$$x_t = t\gamma\phi(x_t) + (I - t\mu F)Tx_t. \tag{23}$$

The following proposition summarizes the basic properties of the net $\{x_t\}$.

Proposition 3.2 *Let $\{x_t\}$ be defined by (23), then*

(i) $\{x_t\}$ *is bounded for* $t \in \big(0, \min\{1, \frac{1}{\tau}\}\big)$.

(ii) $\lim\limits_{t\to 0}\|x_t - Tx_t\| = 0$.

(iii) $\{x_t\}$ *defines a continous curve from* $\big(0, \min\{1, \frac{1}{\tau}\}\big)$ *into* C.

proof *(i) For any $q \in F(T)$, fixed $\epsilon_1 > 0$, for each $t \in \big(0, \min\{1, \frac{1}{\tau}\}\big)$, we have the following cases:*
Case 1. $\|x_t - q\| < \epsilon_1$*; In this case, we can see easily that $\{x_t\}$ is bounded.*

Case 2. $\|x_t - q\| \geq \epsilon_1$. *In this case, by Lemma 2.2, there is a number $r_1 \in (0, 1)$ such that*

$$\|\phi(x_t) - \phi(q)\| \leq r_1\|x_t - q\|.$$

From (23) and Lemma 2.4, we have

$$\begin{aligned}
\|x_t - q\| &= \|t\gamma\phi(x_t) + (I - t\mu F)Tx_t - q\| \\
&= \|t[\gamma\phi(x_t) - \mu Fq] + (I - t\mu F)Tx_t - (I - t\mu F)q\| \\
&\leq \|(I - t\mu F)Tx_t - (I - t\mu F)q\| + t\|\gamma\phi(x_t) - \mu Fq\| \\
&\leq (1 - t\tau)\|x_t - q\| + t\gamma\|\phi(x_t) - \phi(q)\| + t\|\gamma\phi(q) - \mu Fq\| \\
&\leq (1 - t\tau)\|x_t - q\| + t\gamma r_1\|x_t - q\| + t\|\gamma\phi(q) - \mu Fq\| \\
&= [1 - t(\tau - \gamma r_1)]\|x_t - q\| + t\|\gamma\phi(q) - \mu Fq\|.
\end{aligned}$$

It follows that

$$\|x_t - q\| \leq \frac{\|\gamma\phi(q) - \mu Fq\|}{\tau - \gamma r_1}.$$

Hence, $\{x_t\}$ is bounded, so are $\{\phi(x_t)\}$ and $\{FTx_t\}$.

(ii) From (23), we have

$$\|x_t - Tx_t\| = t\|\gamma\phi(x_t) - \mu FTx_t\| \to 0 \quad as \quad t \to 0. \tag{24}$$

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(iii) Take $t, t_0 \in \left(0, \min\{1, \frac{1}{\tau}\}\right)$. *From* (23) *and Lemma 2.4, we have*

$$\begin{aligned}
\|x_t - x_{t_0}\| &= \|[t\gamma\phi(x_t) + (I - t\mu F)Tx_t] - [t_0\gamma\phi(x_{t_0}) + (I - t_0\mu F)Tx_{t_0}]\| \\
&= \|(t - t_0)\gamma\phi(x_t) + t_0\gamma[\phi(x_t) - \phi(x_{t_0})] + (t_0 - t)\mu FTx_t \\
&\quad +(I - t_0\mu F)Tx_t - (I - t_0\mu F)Tx_{t_0}\| \\
&\le (\gamma\|\phi(x_t)\| + \mu\|FTx_t\|)|t - t_0| + t_0\gamma\|x_t - x_{t_0}\| \\
&\quad +(1 - t_0\tau)\|Tx_t - Tx_{t_0}\| \\
&\le (\gamma\|\phi(x_t)\| + \mu\|FTx_t\|)|t - t_0| + [1 - t_0(\tau - \gamma)]\|x_t - x_{t_0}\|.
\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\gamma\|\phi(x_t)\| + \mu\|FTx_t\|}{t_0(\tau - \gamma)}|t - t_0|.$$

This shows that $\{x_t\}$ *is locally Lipschitzian and hence continuous.*

Theorem 3.3 *Assume that* $\{x_t\}$ *is defined by* (23), *then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a fixed point* x^* *of* T *which solves the variational inequality:*

$$\langle(\mu F - \gamma\phi)x^*, x^* - x\rangle \le 0, \quad \forall x \in F(T). \tag{25}$$

proof *We first show the uniqueness of a solution of the variational inequality* (25). *Suppose both* $\overline{x} \in F(T)$ *and* $\overline{y} \in F(T)$ *are solutions to* (25). *Without lost of generality, we may assume there is a positive number* ϵ *such that* $\|\overline{x} - \overline{y}\| \ge \epsilon$. *Then, by Lemma 2.2, there is a number* $r \in (0, 1)$ *such that* $\|\phi(\overline{x}) - \phi(\overline{y})\| \le r\|\overline{x} - \overline{y}\|$.
From (25), *we know*

$$\langle(\mu F - \gamma\phi)\overline{x}, \overline{x} - \overline{y}\rangle \le 0. \tag{26}$$

and

$$\langle(\mu F - \gamma\phi)\overline{y}, \overline{y} - \overline{x}\rangle \le 0. \tag{27}$$

Adding up (26) *and* (27), *we have*

$$\langle(\mu F - \gamma\phi)\overline{x} - (\mu F - \gamma\phi)\overline{y}, \overline{x} - \overline{y}\rangle \le 0.$$

Observe that

$$\begin{aligned}
\frac{\mu L^2}{2} > 0 \quad &\Leftrightarrow \quad \eta - \frac{\mu L^2}{2} < \eta \\
&\Leftrightarrow \quad \mu(\eta - \frac{\mu L^2}{2}) < \mu\eta \\
&\Leftrightarrow \quad \tau < \mu\eta.
\end{aligned}$$

It follows that

$$0 < \gamma < \tau < \mu\eta.$$

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We notice that

$$\begin{aligned}\langle(\mu F-\gamma\phi)\overline{x}-(\mu F-\gamma\phi)\overline{y},\overline{x}-\overline{y}\rangle &= \mu\langle F\overline{x}-F\overline{y},\overline{x}-\overline{y}\rangle-\gamma\langle\phi(\overline{x})-\phi(\overline{y}),\overline{x}-\overline{y}\rangle\\ &\geq \mu\eta\|\overline{x}-\overline{y}\|^2-\gamma\|\phi(\overline{x})-\phi(\overline{y})\|\|\overline{x}-\overline{y}\|\\ &\geq \mu\eta\|\overline{x}-\overline{y}\|^2-\gamma r\|\overline{x}-\overline{y}\|^2\\ &\geq (\mu\eta-\gamma r)\|\overline{x}-\overline{y}\|^2\\ &\geq (\mu\eta-\gamma r)\epsilon^2\\ &> 0.\end{aligned}$$

Therefore, $\overline{x}=\overline{y}$ *and the uniqueness is proved. Below we use* $x^*\in F(T)$ *to denote the unique solution of* (25).

Next, we show that $x_t\to x^*$ *as* $t\to 0$.
Since $\{x_t\}$ *is bounded and* H *is reflexive, there exists a subsequence* $\{x_{t_n}\}$ *of* $\{x_t\}$ *such that* $x_{t_n}\rightharpoonup x^*$. *By* (24), *we have* $x_{t_n}-Tx_{t_n}\to 0$ *as* $t_n\to 0$. *It follows from Lemma 2.3 that* $x^*\in F(T)$.

We claim

$$\|x_{t_n}-x^*\|\to 0.$$

By contradiction, there is a number $\epsilon_0>0$ *and a subsequence* $\{x_{t_m}\}$ *of* $\{x_{t_n}\}$ *such that* $\|x_{t_m}-x^*\|\geq\epsilon_0$. *From Lemma 2.2, there is a number* $r_{\epsilon_0}>0$ *such that* $\|\phi(x_{t_m})-\phi(x^*)\|\leq r_{\epsilon_0}\|x_{t_m}-x^*\|$, *we write*

$$x_{t_m}-x^*=t_m(\gamma\phi(x_{t_m})-\mu Fx^*)+(I-t_m\mu F)Tx_{t_m}-(I-t_m\mu F)x^*,$$

to derive that

$$\begin{aligned}\|x_{t_m}-x^*\|^2 &= t_m\langle\gamma\phi(x_{t_m})-\mu Fx^*,x_{t_m}-x^*\rangle\\ &\quad+\langle(I-t_m\mu F)Tx_{t_m}-(I-t_m\mu F)x^*,x_{t_m}-x^*\rangle\\ &\leq t_m\langle\gamma\phi(x_{t_m})-\mu Fx^*,x_{t_m}-x^*\rangle+(1-t_m\tau)\|x_{t_m}-x^*\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{t_m}-x^*\|^2 &\leq \frac{\langle\gamma\phi(x_{t_m})-\mu Fx^*,x_{t_m}-x^*\rangle}{\tau}\\ &= \frac{\langle\gamma\phi(x_{t_m})-\gamma\phi(x^*),x_{t_m}-x^*\rangle+\langle\gamma\phi(x^*)-\mu Fx^*,x_{t_m}-x^*\rangle}{\tau}\\ &\leq \frac{\gamma r_{\epsilon_0}\|x_{t_m}-x^*\|^2+\langle\gamma\phi(x^*)-\mu Fx^*,x_{t_m}-x^*\rangle}{\tau}.\end{aligned}$$

Therefore,

$$\|x_{t_m}-x^*\|^2\leq\frac{1}{\tau-\gamma r_{\epsilon_0}}\langle\gamma\phi(x^*)-\mu Fx^*,x_{t_m}-x^*\rangle. \tag{28}$$

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From (28)*, we get that* $x_{t_m} \to x^*$*. It is a contradiction. Hence, we have*

$$\|x_{t_n} - x^*\| \to 0.$$

Next, we show that x^* *solves the variational inequality* (25)*. Since*

$$x_t = t\gamma\phi(x_t) + (I - t\mu F)Tx_t,$$

we derive that

$$(\mu F - \gamma\phi)x_t = -\frac{1}{t}(I - T)x_t + \mu(Fx_t - FTx_t). \tag{29}$$

But, $(I - T)$ *is accretive, that is;*

$$\begin{aligned}
\langle (I-T)x_t - (I-T)z, x_t - z\rangle &\geq \|x_t - z\|^2 - \|Tx_t - Tz\|\|x_t - z\| \\
&\geq \|x_t - z\|^2 - \|x_t - z\|^2 \\
&= 0.
\end{aligned}$$

It follows from (29) *that, for all* $z \in F(T)$,

$$\begin{aligned}
\langle (\mu F - \gamma\phi)x_t, x_t - z\rangle &= -\frac{1}{t}\langle (I-T)x_t - (I-T)z, x_t - z\rangle + \mu\langle Fx_t - FTx_t, x_t - z\rangle \\
&\leq \mu\langle Fx_t - FTx_t, x_t - z\rangle \\
&\leq \mu L\|x_t - Tx_t\|\|x_t - z\| \\
&\leq \|x_t - Tx_t\|M, \qquad (30)
\end{aligned}$$

where M *is an appropriate constant such that* $M = \sup\{\mu L\|x_t - z\|\}$*, where* $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$*. Now, replacing* t *in* (30) *with* t_n *and letting* $n \to \infty$*, noticing that* $(I - T)x_{t_n} \to (I - T)x^* = 0$ *for* $x^* \in F(T)$*, we obtain*

$$\langle (\mu F - \gamma\phi)x^*, x^* - z\rangle \leq 0.$$

That is, $x^* \in F(T)$ *is a solution of the variational inequality* (25)*. Hence,* $x^* = \overline{x}$ *by uniqueness. We have shown that each cluster point of* $\{x_t\}$ *(at* $t \to 0$*) equals* $\overline{x}$*. Therefore,*

$$\lim_{t\to 0}\|x_t - \overline{x}\| = 0.$$

This completes the proof.

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Theorem 3.4 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \to C$ be an η-strongly monotone and L-Lipschitzian operator with $\eta > 0$, $L > 0$ and let ϕ be an MKC on C. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable infinite family of nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\{\lambda_i\}_{i=1}^{\infty}$ be a real sequence such that $0 < \lambda_i \leq b < 1$ for all $i \in \mathbb{N}$, for some constant $b \in (0,1)$. Suppose $W_n : C \to C$ be given by (21) for all $n \in \mathbb{N}$. Assume that $0 < \mu < 2\eta/L^2$ and $0 < \gamma < \tau$, where $\tau = \mu(\eta - \frac{\mu L^2}{2})$. Given that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$

$(C2)$ $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

$(C3)$ $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Let $\{x_n\}$ be the sequence generated by the iterative scheme (22), then $\{x_n\}$ converges strongly to a fixed point x^* of $\{T_i\}_{i=1}^{\infty}$, which solves the variational inequality:

$$\langle (\gamma\phi - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i).$$

proof *First, we start by showing that the sequence $\{x_n\}$ and $\{y_n\}$ are bounded. Indeed, take a point $q \in \bigcap_{i=1}^{\infty} F(T_i)$, we notice that*

$$\begin{aligned} \|y_n - q\| &= \|\beta_n x_n + (1-\beta_n)W_n x_n - q\| \\ &= \|\beta_n x_n + \beta_n q - \beta_n q + W_n x_n - \beta_n W_n x_n - q\| \\ &\leq \beta_n \|x_n - q\| + (1-\beta_n)\|W_n x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Without loss of generality, we can assume $\alpha_n \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$.
From definition of MKC and Lemma 2.2, for any $\epsilon > 0$, there is a number $r_\epsilon \in (0,1)$ such that:
(i) if $\|x_n - q\| < \epsilon$ then $\|\phi(x_n) - \phi(q)\| < \epsilon$;

(ii) if $\|x_n - q\| \geq \epsilon$ then $\|\phi(x_n) - \phi(q)\| \leq r_\epsilon \|x_n - q\|$.

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It follows from (22) *and Lemma 2.4 that*

$$\begin{aligned}
\|x_{n+1}-q\| &= \|P_C[\alpha_n\gamma\phi(x_n)+(I-\alpha_n\mu F)y_n]-P_C(q)\| \\
&\le \|\alpha_n\gamma\phi(x_n)+(I-\alpha_n\mu F)y_n-q\| \\
&= \|\alpha_n[\gamma\phi(x_n)-\mu Fq]+(I-\alpha_n\mu F)y_n-(I-\alpha_n\mu F)q\| \\
&\le \|(I-\alpha_n\mu F)y_n-(I-\alpha_n\mu F)q\|+\alpha_n\|\gamma\phi(x_n)-\mu Fq\| \\
&\le (1-\alpha_n\tau)\|y_n-q\|+\alpha_n\|\gamma\phi(x_n)-\gamma\phi(q)+\gamma\phi(q)-\mu Fq\| \\
&\le (1-\alpha_n\tau)\|x_n-q\|+\alpha_n\gamma\|\phi(x_n)-\phi(q)\|+\alpha_n\|\gamma\phi(q)-\mu Fq\| \\
&= (1-\alpha_n\tau)\|x_n-q\|+\alpha_n\gamma\max\{r_\epsilon\|x_n-q\|,\epsilon\}+\alpha_n\|\gamma\phi(q)-\mu Fq\| \\
&= \max\{(1-\alpha_n\tau)\|x_n-q\|+\alpha_n\gamma r_\epsilon\|x_n-q\|+\alpha_n\|\gamma\phi(q)-\mu Fq\|, \\
&\qquad (1-\alpha_n\tau)\|x_n-q\|+\alpha_n\gamma\epsilon+\alpha_n\|\gamma\phi(q)-\mu Fq\|\} \\
&= \max\{(1-\alpha_n(\tau-\gamma r_\epsilon))\|x_n-q\|+\alpha_n\|\gamma\phi(q)-\mu Fq\|, \\
&\qquad (1-\alpha_n\tau)\|x_n-q\|+\alpha_n\gamma\epsilon+\alpha_n\|\gamma\phi(q)-\mu Fq\|\} \\
&= \max\Big\{(1-\alpha_n(\tau-\gamma r_\epsilon))\|x_n-q\|+\alpha_n(\tau-\gamma r_\epsilon)\frac{\|\gamma\phi(q)-\mu Fq\|}{\tau-\gamma r_\epsilon}, \\
&\qquad (1-\alpha_n\tau)\|x_n-q\|+\alpha_n(\tau-\gamma r_\epsilon)\frac{\gamma\epsilon+\|\gamma\phi(q)-\mu Fq\|}{\tau-\gamma r_\epsilon}\Big\} \\
&\le \max\left\{\|x_n-q\|,\frac{\gamma\epsilon+\|\gamma\phi(q)-\mu Fq\|}{\tau-\gamma r_\epsilon}\right\}
\end{aligned}$$

By simple inductions, we have

$$\|x_n-q\|\le\max\left\{\|x_0-q\|,\frac{\gamma\epsilon+\|\gamma\phi(q)-\mu Fq\|}{\tau-\gamma r_\epsilon}\right\},\quad n\ge 0.$$

Hence $\{x_n\}$ *is bounded, and so are* $\{y_n\}$, $\{Fy_n\}$ *and* $\{\phi(x_n)\}$.

Since W_n *is nonexpansive and* $y_n=\beta_nx_n+(1-\beta_n)W_nx_n$, *we also have*

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|[\beta_{n+1}x_{n+1}+(1-\beta_{n+1})W_{n+1}x_{n+1}]-[\beta_nx_n+(1-\beta_n)W_nx_n]\| \\
&= \|\beta_{n+1}x_{n+1}-\beta_{n+1}x_n+\beta_{n+1}x_n-\beta_nx_n+(1-\beta_{n+1})(W_{n+1}x_{n+1}-W_{n+1}x_n) \\
&\quad +(1-\beta_{n+1})(W_{n+1}x_n-W_nx_n)+(1-\beta_{n+1})W_nx_n-(1-\beta_n)W_nx_n\| \\
&\le \beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|+(1-\beta_{n+1})\|W_{n+1}x_{n+1}-W_{n+1}x_n\| \\
&\quad +(1-\beta_{n+1})\|W_{n+1}x_n-W_nx_n\|+|\beta_n-\beta_{n+1}|\|W_nx_n\| \\
&\le \beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|+(1-\beta_{n+1})\|x_{n+1}-x_n\| \\
&\quad +(1-\beta_{n+1})\|W_{n+1}x_n-W_nx_n\|+|\beta_n-\beta_{n+1}|\|W_nx_n\| \\
&= \|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|+(1-\beta_{n+1})\|W_{n+1}x_n-W_nx_n\| \\
&\quad +|\beta_n-\beta_{n+1}|\|W_nx_n\|. \qquad (31)
\end{aligned}$$

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By using the Lemma 2.6, we can conclude that

$$\|W_n x_{n-1} - W_{n-1}x_{n-1}\| \le 2\prod_{i=1}^{n}\lambda_i\|x_{n-1}-q\| \to 0 \ as \ n\to\infty. \tag{32}$$

From (22)*, the inequalities* (31) *and* (32)*, we have*

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \|[\alpha_n\gamma\phi(x_n)+(I-\alpha_n\mu F)y_n]-[\alpha_{n-1}\gamma\phi(x_{n-1})+(I-\alpha_{n-1}\mu F)y_{n-1}]\| \\
&= \|[\alpha_n\gamma\phi(x_n)-\alpha_n\gamma\phi(x_{n-1})]+[\alpha_n\gamma\phi(x_{n-1})-\alpha_{n-1}\gamma\phi(x_{n-1})] \\
&\quad +[(I-\alpha_n\mu F)y_n-(I-\alpha_n\mu F)y_{n-1}]+[\alpha_{n-1}\mu Fy_{n-1}-\alpha_n\mu Fy_{n-1}]\| \\
&= \|\alpha_n\gamma[\phi(x_n)-\phi(x_{n-1})]+[(I-\alpha_n\mu F)y_n-(I-\alpha_n\mu F)y_{n-1}] \\
&\quad +(\alpha_n-\alpha_{n-1})\gamma\phi(x_{n-1})+(\alpha_{n-1}-\alpha_n)\mu Fy_{n-1}\| \\
&\le \alpha_n\gamma\|\phi(x_n)-\phi(x_{n-1})\|+\|(I-\alpha_n F)y_n-(I-\alpha_n\mu F)y_{n-1}\| \\
&\quad +|\alpha_n-\alpha_{n-1}|\|\gamma\phi(x_{n-1})+\mu Fy_{n-1}\| \\
&\le \alpha_n\gamma\|x_n-x_{n-1}\|+(1-\alpha_n\tau)\|y_n-y_{n-1}\| \\
&\quad +|\alpha_n-\alpha_{n-1}|(\gamma\|\phi(x_{n-1})\|+\mu\|Fy_{n-1}\|) \\
&\le \alpha_n\gamma\|x_n-x_{n-1}\|+(1-\alpha_n\tau)[\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad +(1-\beta_n)\|W_nx_{n-1}-W_{n-1}x_{n-1}\|+|\beta_{n-1}-\beta_n|\|W_{n-1}x_{n-1}\|] \\
&\quad +|\alpha_n-\alpha_{n-1}|M_1 \\
&\le [1-\alpha_n(\tau-\gamma)]\|x_n-x_{n-1}\|+M_2|\beta_n-\beta_{n-1}| \\
&\quad +2(1-\beta_n)\prod_{i=1}^{n}\lambda_i\|x_{n-1}-q\|+|\alpha_n-\alpha_{n-1}|M_1 \\
&\le [1-\alpha_n(\tau-\gamma)]\|x_n-x_{n-1}\|+M_3(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|) \\
&\quad +2(1-\beta_n)\prod_{i=1}^{n}\lambda_i\|x_{n-1}-q\|,
\end{aligned}$$

where M_1, M_2, *and* M_3 *are appropriate constants such that* $M_3 \ge \max\{M_1, M_2\}$*:* $M_1 \ge \sup_{n\ge1}\{\gamma\|\phi(x_n)\|+\|\mu Fy_n\|\}$, $M_2 \ge \sup_{n\ge1}\{\|x_n\|+\|W_nx_n\|\}$.

From the condition $(C3)$ *and Lemma 2.1, we obtain that*

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{33}$$

From (22)*, we also have*

$$\begin{aligned}
\|x_{n+1}-y_n\| &= \|P_C[\alpha_n\gamma\phi(x_n)+(I-\alpha_n\mu F)y_n]-P_C(y_n)\| \\
&\le \|\alpha_n\gamma\phi(x_n)+(I-\alpha_n\mu F)y_n-y_n\| \\
&\le \alpha_n\|\gamma\phi(x_n)\|+\alpha_n\|\mu F(y_n)\| \\
&= \alpha_n\big(\|\gamma\phi(x_n)\|+\|\mu Fy_n\|\big).
\end{aligned}$$

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Using the condition $(C1)$, *we obtain*

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{34}$$

Since

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|,$$

using (33) *and* (34), *we have that*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{35}$$

On the other hand, we note that

$$\begin{aligned} \|W_n x_n - x_n\| &\le \|x_n - y_n\| + \|y_n - W_n x_n\| \\ &= \|x_n - y_n\| + \|(\beta_n x_n + (1-\beta_n) W_n x_n) - W_n x_n\| \\ &= \|x_n - y_n\| + \beta_n \|x_n - W_n x_n\|, \end{aligned}$$

which implies that

$$\|W_n x_n - x_n\| \le \frac{1}{1-\beta_n} \|x_n - y_n\|.$$

From the condition $(C2)$ *and* (35), *we obtain*

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{36}$$

Next, we show that $\limsup_{n\to\infty} \langle (\gamma\phi - \mu F)x^*, x_n - x^* \rangle \le 0$, *where* $x^* = \lim_{t\to 0} x_t$ *with* x_t *being the fixed point of the contraction*

$$x \mapsto t\gamma\phi(x) + (I - t\mu F) W_n x.$$

Thus, x_t, *by Lemma 3.1, solves the fixed point equation*

$$x_t = t\gamma\phi(x_t) + (I - t\mu F) W_n x_t.$$

To show this, we take a subsequence $\{x_{n_k}\}$ *of the sequence* $\{x_n\}$ *such that*

$$\limsup_{n\to\infty} \langle (\gamma\phi - \mu F)x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle (\gamma\phi - \mu F)x^*, x_{n_k} - x^* \rangle.$$

We may also assume that $x_{n_k} \rightharpoonup q$. *Note that* $q \in F(W_n) = F(T)$ *in virtue of Lemmas 2.3, 2.5 and* (36). *It follows from Theorem 3.3, We can get that*

$$\begin{aligned} \limsup_{n\to\infty} \langle (\gamma\phi - \mu F)x^*, x_n - x^* \rangle &= \lim_{k\to\infty} \langle (\gamma\phi - \mu F)x^*, x_{n_k} - x^* \rangle \\ &= \langle (\gamma\phi - \mu F)x^*, q - x^* \rangle \le 0. \end{aligned}$$

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Hence

$$\limsup_{n\to\infty}\langle(\gamma\phi-\mu F)x^*, x_n - x^*\rangle \le 0. \tag{37}$$

Finally, we show that $\lim_{n\to\infty}\|x_n - x^*\| = 0.$

By contradiction, there is a number $\epsilon_0 > 0$ *such that*

$$\limsup_{n\to\infty}\|x_n - x^*\| \ge \epsilon_0.$$

Case 1. *Fixed* ϵ_1 $(\epsilon_1 < \epsilon_0)$*, if for some* $n \ge N \in \mathbb{N}$ *such that* $\|x_n - x^*\| \ge \epsilon_0 - \epsilon_1$*, and for the other* $n \ge N \in \mathbb{N}$ *such that* $\|x_n - x^*\| < \epsilon_0 - \epsilon_1$.

Let

$$M_n = \frac{2\langle(\gamma\phi-\mu F)x^*, x_{n+1}-x^*\rangle}{(\epsilon_0-\epsilon_1)^2}.$$

From (37)*, we know* $\limsup_{n\to\infty} M_n \le 0$*. Hence, there are two numbers* h *and* N*, when* $n > N$ *we have* $M_n \le h$*, where* $h = \min\{\tau-\gamma\}$*. From the above introduction, we can extract a number* $n_0 > N$ *satisfying* $\|x_{n_0} - x^*\| < \epsilon_0 - \epsilon_1$*, then we estimate* $\|x_{n_0+1} - x^*\|$*. From* (22) *and Lemma 2.4, we have*

$$\begin{aligned}
\|x_{n_0+1}-x^*\|^2 &= \|P_C[\alpha_{n_0}\gamma\phi(x_{n_0}) + (I-\alpha_{n_0}\mu F)y_{n_0}] - P_C(x^*)\|^2\\
&\le \|\alpha_{n_0}\gamma\phi(x_{n_0}) + (I-\alpha_{n_0}\mu F)y_{n_0} - x^*\|^2\\
&= \|\alpha_{n_0}[\gamma\phi(x_{n_0}) - \mu F x^*] + (I-\alpha_{n_0}\mu F)y_{n_0} - (I-\alpha_{n_0}\mu F)x^*\|^2\\
&= \langle(I-\alpha_{n_0}\mu F)y_{n_0} - (I-\alpha_{n_0}\mu F)x^*, x_{n_0+1}-x^*\rangle\\
&\quad + \alpha_{n_0}\gamma\langle\phi(x_{n_0}) - \phi(x^*), x_{n_0+1}-x^*\rangle + \alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle\\
&\le \|(I-\alpha_{n_0}\mu F)y_{n_0} - (I-\alpha_{n_0}\mu F)x^*\|\|x_{n_0+1}-x^*\|\\
&\quad + \alpha_{n_0}\gamma\|x_{n_0}-x^*\|\|x_{n_0+1}-x^*\| + \alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle\\
&\le (1-\alpha_{n_0}\tau)\|y_{n_0}-x^*\|\|x_{n_0+1}-x^*\| + \alpha_{n_0}\gamma\|x_{n_0}-x^*\|\|x_{n_0+1}-x^*\|\\
&\quad + \alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle\\
&\le [1-\alpha_{n_0}(\tau-\gamma)]\|x_{n_0}-x^*\|\|x_{n_0+1}-x^*\|\\
&\quad + \alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle\\
&\le \frac{1}{2}[1-\alpha_{n_0}(\tau-\gamma)]\|x_{n_0}-x^*\|^2 + \frac{1}{2}\|x_{n_0+1}-x^*\|^2\\
&\quad + \alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle\\
&< \frac{1}{2}[1-\alpha_{n_0}(\tau-\gamma)](\epsilon_0-\epsilon_1)^2 + \frac{1}{2}\|x_{n_0+1}-x^*\|^2\\
&\quad + \alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n_0+1}-x^*\|^2 &< [1-\alpha_{n_0}(\tau-\gamma)](\epsilon_0-\epsilon_1)^2 + 2\alpha_{n_0}\langle\gamma\phi(x^*) - \mu F x^*, x_{n_0+1}-x^*\rangle\\
&= [1-\alpha_{n_0}(\tau-\gamma-M_{n_0})](\epsilon_0-\epsilon_1)^2\\
&\le (\epsilon_0-\epsilon_1)^2.
\end{aligned}$$

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Hence, we have

$$\|x_{n_0+1} - x^*\| \leq \epsilon_0 - \epsilon_1.$$

In the same way, we can get

$$\|x_n - x^*\| < \epsilon_0 - \epsilon_1, \quad \forall n > n_0.$$

It contradict the $\limsup_{n\to\infty}\|x_n - x^*\| \geq \epsilon_0$.

Case 2. *Fixed* ϵ_1 $(\epsilon_1 < \epsilon_0)$, *if* $\|x_n - x^*\| \geq \epsilon_0 - \epsilon_1$, *for all* $n \geq N \in \mathbb{N}$. *In this case from Lemma 2.2, there is a number* $r \in (0,1)$, *such that*

$$\|\phi(x_n) - \phi(x^*)\| \leq r\|x_n - x^*\|, \quad n \geq N.$$

From (22) *and Lemma 2.4, we have that*

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_C[\alpha_n\gamma\phi(x_n) + (I - \alpha_n\mu F)y_n] - P_C(x^*)\|^2 \\
&\leq \|\alpha_n\gamma\phi(x_n) + (I - \alpha_n\mu F)y_n - x^*\|^2 \\
&= \|\alpha_n[\gamma\phi(x_n) - \mu F x^*] + (I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)x^*\|^2 \\
&= \langle (I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n\gamma\langle \phi(x_n) - \phi(x^*), x_{n+1} - x^*\rangle + \alpha_n\langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\leq \|(I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)x^*\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n\gamma r\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n\tau)\|y_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\gamma r\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n\langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma r)]\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\leq \frac{1}{2}[1 - \alpha_n(\tau - \gamma r)]\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 \\
&\quad + \alpha_n\langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq [1 - \alpha_n(\tau - \gamma r)]\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma\phi(x^*) - F x^*, x_{n+1} - x^*\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma r)]\|x_n - x^*\|^2 \\
&\quad + \alpha_n(\tau - \gamma r)\frac{2\langle \gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle}{\tau - \gamma r}. \qquad (38)
\end{aligned}$$

Applying Lemma 2.1 to (38), *we conclude that* $x_n \to x^*$ *as* $n \to \infty$. *It also contradict the*

$$\|x_n - x^*\| \geq \epsilon_0 - \epsilon_1.$$

This completes the proof.

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4. Conclusion

We conclude the article with the following observations.

(i) Theorem 3.4 improve and extend Theorem 3.2 of Kim and Xu [9], Theorem 1 of Yao et al. [23], Theorem 3.4 of Marino and Xu [5], Theorem 3.2 of Tian[20], Theorem 2.1 of Shang et al. [13], Theorem 2.1 of Singthong and Suantai[15] and includes those results as special cases. Especially, our results extend above results from contractions to more general MKC. Our iterative scheme studied in this article can be viewed as a refinement and modification of the iterative methods in [5, 9, 13, 15, 20, 21, 23]. On the other hand, our iterative schemes concern a countable infinite family of nonexpansive mappings, in this respect, they can be viewed as another improvement.

(ii) Our results extend the results of; Marino and Xu [5], Shang et al. [13], Singthong and Suantai [15], from strong positive linear bounded operator to η-strongly monotone and L-Lipschitzian operator.

(iii) The advantage of the results in this paper is that less restrictions on the parameters $\{\gamma_{n,i}\}$ in [13, 15] are imposed. Our results unify many recent results including the results in [5, 9, 13, 15, 20, 21, 23].

(iv) It is worth noting that we obtained strong convergence result concerning a countable infinite family of nonexpansive mappings. Our result is new and the proofs are simple and different from those in [5, 9, 13, 15, 20, 21, 23].

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