

# Synchronal Algorithm For a Countable Family of Strict Pseudocontractions in $q$ -uniformly Smooth Banach Spaces

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## Abstract

Let  $E$  be a real  $q$ -uniformly smooth Banach space whose duality map is weakly sequentially continuous and  $C$  be a nonempty, closed and convex subset of  $E$ . Let  $\{T_i\}_{i=1}^{\infty} : C \rightarrow E$  be a family of  $k$ -strict pseudocontractions for  $k \in (0, 1)$  such that  $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ ,  $f$  be a contraction with coefficient  $\beta \in (0, 1)$  and  $\{\lambda_i\}_{i=1}^{\infty}$  be a real sequence in  $(0, 1)$  such that  $\sum_{i=1}^{\infty} \lambda_i = 1$ . Let  $G : C \rightarrow E$  be an  $\eta$ -strongly accretive and  $L$ -Lipschitzian operator with  $L > 0$ ,  $\eta > 0$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $(0, 1)$  satisfying some conditions. For some positive real numbers  $\gamma$ ,  $\mu$  appropriately chosen, let  $\{x_n\}$  be a sequence defined by

$$\begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ T^{\beta_n} = \beta_n I + (1 - \beta_n) \sum_{i=1}^{\infty} \lambda_i T_i, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T^{\beta_n} x_n, \quad n \geq 0. \end{cases}$$

Then, we prove that  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of the countable family  $\{T_i\}_{i=1}^{\infty}$ , which solves the variational inequality:

$$\langle (\gamma f - \mu G)x^*, j_q(x - x^*) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i).$$

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# 1 Introduction

Let  $E$  be a real Banach space and  $E^*$  be the dual of  $E$ . For some real number  $q$  ( $1 < q < \infty$ ), the *generalized duality mapping*  $J_q : E \rightarrow 2^{E^*}$  is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E, \quad (1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between elements of  $E$  and those of  $E^*$ . In particular,  $J = J_2$  is called the *normalized duality mapping* and  $J_q(x) = \|x\|^{q-2} J_2(x)$  for  $x \neq 0$ . If  $E$  is a real Hilbert space, then  $J = I$ , where  $I$  is the identity mapping. It is well known that if  $E$  is smooth, then  $J_q$  is single-valued, which is denoted by  $j_q$  (see [16]). The duality mapping  $J_q$  from a smooth Banach space  $E$  into  $E^*$  is said to be *weakly sequentially continuous generalized duality mapping* if for all  $\{x_n\} \subset E$  with  $x_n \rightarrow x$  implies  $J_q(x_n) \rightarrow^* J_q(x)$ .

Let  $C$  be a nonempty closed convex subset of  $E$ , and  $G : E \rightarrow E$  be a nonlinear map. Then, a *variational inequality problem* with respect to  $C$  and  $G$  is to find a point  $x^* \in C$  such that

$$\langle Gx^*, j_q(x - x^*) \rangle \geq 0, \quad \forall x \in C \text{ and } j_q(x - x^*) \in J_q(x - x^*). \quad (2)$$

We denote by  $VI(G, C)$  the set of solutions of this variational inequality problem.

If  $E = H$ , a real Hilbert space, the variational inequality problem reduces to the following: Find a point  $x^* \in C$  such that

$$\langle Gx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (3)$$

A mapping  $T : E \rightarrow E$  is said to be *L-Lipschitzian* if there exists  $L > 0$  such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in E. \quad (4)$$

If  $L = 1$ , then  $T$  is called *Nonexpansive* and if  $0 \leq L < 1$ ,  $T$  is called *Contraction*.

A point  $x \in E$  is called a fixed point of the map  $T$  if  $Tx = x$ . We denote by  $F(T)$  the set of all fixed points of the mapping  $T$ , that is

$$F(T) = \{x \in C : Tx = x\}.$$

We assume that  $F(T) \neq \emptyset$  in the sequel. It is well known that  $F(T)$  above, is closed and convex (see e.g. Goebel and Kirk [7]).

An Operator  $F : E \rightarrow E$  is said to be *Accretive* if  $\forall x, y \in E$ , there exists  $j_q(x - y) \in J_q(x - y)$  such that

$$\langle Fx - Fy, j_q(x - y) \rangle \geq 0. \quad (5)$$

For some positive real numbers  $\eta, \lambda$ , the mapping  $F$  is said to be  *$\eta$ -strongly accretive* if for any  $x, y \in E$ , there exists  $j_q(x - y) \in J_q(x - y)$  such that

$$\langle Fx - Fy, j_q(x - y) \rangle \geq \eta\|x - y\|^q, \quad (6)$$

and it is called  $\lambda$ -strictly pseudocontractive if

$$\langle Fx - Fy, j_q(x - y) \rangle \leq \|x - y\|^q - \lambda \|x - y - (Fx - Fy)\|^q. \quad (7)$$

It is clear that (7) is equivalent to the following

$$\langle (I - F)x - (I - F)y, j_q(x - y) \rangle \geq \lambda \|x - y - (Fx - Fy)\|^q, \quad (8)$$

where  $I$  denotes the identity operator.

In Hilbert spaces, accretive operators are called *monotone* where inequality (5) holds with  $j_q$  replace by identity map of  $H$ .

The modulus of smoothness of  $E$ , with  $\dim E \geq 2$ , is a function  $\rho_E : [0, \infty) \rightarrow [0, \infty)$  defined by

$$\rho_E(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq \tau \right\}.$$

A Banach space  $E$  is said to be uniformly smooth if  $\lim_{t \rightarrow 0^+} \frac{\rho_E(t)}{t} = 0$ . For  $q > 1$ , a Banach space  $E$  is said to be  $q$ -uniformly smooth, if there exists a fixed constant  $c > 0$  such that  $\rho_E(t) \leq ct^q$ ,  $t > 0$ .

It is well known (see [5]) that Hilbert spaces and  $L^p$  ( $p > 1$ ) spaces are uniformly smooth. More precisely,

$$L^p \text{ (or } l_p) \text{ spaces are } \begin{cases} 2\text{-uniformly smooth, if } 2 \leq p < \infty, \\ p\text{-uniformly smooth, if } 1 < p \leq 2. \end{cases}$$

Also, Every  $l_p$  space, ( $1 < p < \infty$ ) has a weakly sequentially continuous duality map.

Let  $K$  be a nonempty closed convex and bounded subset of a Banach space  $E$  and let the diameter of  $K$  be defined by  $d(K) := \sup\{\|x - y\| : x, y \in K\}$ . For each  $x \in K$ , let  $r(x, K) := \sup\{\|x - y\| : y \in K\}$  and let  $r(K) := \inf\{r(x, K) : x \in K\}$  denote the Chebyshev radius of  $K$  relative to itself. The normal structure coefficient  $N(E)$  of  $E$  (see, e.g., [3]) is defined by  $N(E) := \inf\left\{\frac{d(K)}{r(K)} : d(K) > 0\right\}$ . A space  $E$  such that  $N(E) > 1$  is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [6, 9]).

Let  $\mu$  be a continuous linear functional on  $l^\infty$  and  $(a_0, a_1, \dots) \in l^\infty$ . We write  $\mu_n(a_n)$  instead of  $\mu((a_0, a_1, \dots))$ . We call  $\mu$  a Banach limit if  $\mu$  satisfies  $\|\mu\| = \mu_n(1) = 1$  and  $\mu_n(a_{n+1}) = \mu_n(a_n)$  for all  $(a_0, a_1, \dots) \in l^\infty$ . If  $\mu$  is a Banach limit, then

$$\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$$

for all  $(a_0, a_1, \dots) \in l^\infty$ . (see, e.g., [5, 6]).

The Variational inequality problem was initially introduced and studied by Stampacchia [14] in 1964. In the recent years, variational inequality problems have been extended to study a large variety of problems arising in structural analysis, economics and optimization. Thus, the problem of solving a variational inequality of the form (2) has

been intensively studied by numerous authors (see for example, [10, 17, 18, 20] and the references therein).

Let  $H$  be a real Hilbert space. In 2001, Yamada [20] proposed a hybrid steepest descent method for solving variational inequality as follows; Let  $x_0 \in H$  be chosen arbitrary and define a sequence  $\{x_n\}$  by

$$x_{n+1} = Tx_n - \mu\alpha_n F(Tx_n), \quad n \geq 0, \quad (9)$$

where  $T$  is a nonexpansive mapping on  $H$ ,  $F$  is  $L$ -Lipschitzian and  $\eta$ -strongly monotone with  $L > 0$ ,  $\eta > 0$ ,  $0 < \mu < 2\eta/L^2$ . If  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying the following conditions:

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$$

$$(C3) \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then he prove that the sequence  $\{x_n\}$  converges strongly to the unique solution of the variational inequality:

$$\langle F\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T). \quad (10)$$

In 2006, Marino and Xu [10] considered the following general iterative method: starting with an arbitrary initial point  $x_0 \in H$ , define a sequence  $\{x_n\}$  by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \quad (11)$$

where  $T$  is a nonexpansive mapping of  $H$ ,  $f$  is a contraction,  $A$  is a linear bounded strongly positive operator, and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying the conditions (C1) – (C3). They proved that the sequence  $\{x_n\}$  converges strongly to a fixed point  $\tilde{x}$  of  $T$  which solves the variational inequality:

$$\langle (\gamma f - A)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in F(T). \quad (12)$$

In 2010, Tian [17] combined the iterative method (11) with that of Yamada's (9) and considered the following general iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \geq 0, \quad (13)$$

where  $T$  is a nonexpansive mapping on  $H$ ,  $f$  is a contraction,  $F$  is  $k$ -Lipschitzian and  $\eta$ -strongly monotone with  $k > 0$ ,  $\eta > 0$ ,  $0 < \mu < 2\eta/k^2$ . He proved that if the sequence  $\{\alpha_n\}$  of parameters satisfies conditions (C1) – (C3), then the sequence  $\{x_n\}$  generated by (13) converges strongly to a fixed point  $\tilde{x}$  of  $T$  which solves the variational inequality:

$$\langle (\gamma f - \mu F)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in F(T). \quad (14)$$

Very recently, in 2011, Tian and Di [18] studied an algorithm, based on Tian [17] general Iterative algorithm, and proved the following theorem:

**Theorem 1.1** (Synchronal Algorithm)

Let  $H$  be a real Hilbert space and Let  $T_i : H \rightarrow H$  be a  $k_i$ -strictly pseudocontractions for some  $k_i \in (0, 1)$  such that  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ , and  $f$  be a contraction with coefficient  $\beta \in (0, 1)$  and  $\lambda_i$  be a positive constants such that  $\sum_{i=1}^N \lambda_i = 1$ . Let  $G : H \rightarrow H$  be an  $\eta$ -strongly monotone and  $L$ -Lipschitzian operator with  $L > 0, \eta > 0$ . Assume that  $0 < \mu < 2\eta/L^2, 0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\beta = \tau/\beta$ . Let  $x_0 \in H$  be chosen arbitrarily and let  $\{\alpha_n\}, \{\beta_n\}$  be sequences in  $(0, 1)$ , satisfying the following conditions:

- (T1)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty;$
- (T2)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$
- (T3)  $0 < \max k_i \leq \beta_n < a < 1, \forall n \geq 0.$

Let  $\{x_n\}$  be a sequences defined by the composite process

$$\begin{cases} T^{\beta_n} = \beta_n I + (1 - \beta_n) \sum_{i=1}^N \lambda_i T_i, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T^{\beta_n} x_n, \quad n \geq 0. \end{cases} \tag{15}$$

Then  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of  $\{T_i\}_{i=1}^N$  which solves the variational inequality:

$$\langle (\gamma f - \mu G)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^N F(T_i). \tag{16}$$

The following questions naturally arise in connection with above results:

**Question 1.** Can Theorem of Tian and Di [18] be extend from a real Hilbert space to a general Banach space? such as  $q$ -uniformly smooth Banach space.

**Question 2.** Can we extend the iterative method of scheme (15) to a general iterative scheme define over the set of fixed points of a countable infinite family of strict pseudocontractions.

The purpose of this paper is to give the affirmative answers to these questions mentioned above.

Throughout this paper, we will use the following notations:

1.  $\rightharpoonup$  for weak convergence and  $\rightarrow$  for strong convergence.
2.  $\omega_\omega(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$  denotes the weak  $\omega$ -limit set of  $\{x_n\}$ .

## 2 Preliminaries

In the sequel, we shall make use of the following lemmas.

**Lemma 2.1** (Boonchari and Saejung, [1, 2]) *Let  $C$  be a nonempty, closed and convex subset of a smooth Banach space  $E$ . Suppose that  $\{T_i\}_{i=1}^\infty : C \rightarrow E$  is a family of  $\lambda$ -strictly pseudocontractive mappings with  $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$  and  $\{\mu_i\}_{i=1}^\infty$  is a real sequence in  $(0, 1)$  such that  $\sum_{i=1}^\infty \mu_i = 1$ . Then the following conclusions hold:*

- (i) *A mapping  $G : C \rightarrow E$  defined by  $G := \sum_{i=1}^\infty \mu_i T_i$  is a  $\lambda$ -strictly pseudocontractive*

mapping.

$$(ii) \quad F(G) = \bigcap_{i=1}^{\infty} F(T_i).$$

**Lemma 2.2** (Lim and Xu, [9]) *Suppose  $E$  is a Banach space with uniform normal structure,  $K$  is a nonempty bounded subset of  $E$ , and  $T : K \rightarrow K$  is uniformly  $k$ -Lipschitzian mapping with  $k < N(E)^{\frac{1}{2}}$ . Suppose also there exists nonempty bounded closed convex subset  $C$  of  $K$  with the following property (P) : (P)  $x \in C$  implies  $\omega_{\omega}(x) \subset C$ , where  $\omega_{\omega}(x)$  is the  $\omega$ -limit set of  $T$  at  $x$ , i.e., the set*

$$\{y \in E : y = \text{weak} - \lim_j T^{n_j} x \text{ for some } n_j \rightarrow \infty\}.$$

Then  $T$  has a fixed point in  $C$ .

**Lemma 2.3** (Sunthrayuth and Kumam, [15]) *Let  $C$  be a nonempty, closed and convex subset of a real  $q$ -uniformly smooth Banach space  $E$  which admits a weakly sequentially continuous generalized duality mapping  $j_q$  from  $E$  into  $E^*$ . Let  $T : C \rightarrow C$  be a nonexpansive mapping with  $F(T) \neq \emptyset$ . Then, for all  $\{x_n\} \subset C$ , if  $x_n \rightarrow x$  and  $x_n - Tx_n \rightarrow 0$ , then  $x = Tx$ .*

**Lemma 2.4** (Petryshyn, [12]) *Let  $E$  be a real  $q$ -uniformly smooth Banach space and let  $J_q : E \rightarrow 2^{E^*}$  be the generalized duality mapping. Then for any  $x, y \in E$  and  $j_q(x + y) \in J_q(x + y)$ ,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y) \rangle.$$

**Lemma 2.5** (Sunthrayuth and Kumam, [15]) *Let  $C$  be a nonempty, closed and convex subset of a real  $q$ -uniformly smooth Banach space  $E$ . Let  $F : C \rightarrow E$  be a  $\eta$ -strongly accretive and  $L$ -Lipschitzian operator with  $\eta > 0$ ,  $L > 0$ . Assume that  $0 < \mu < \left(\frac{q\eta}{d_q L^q}\right)^{\frac{1}{q-1}}$  and  $\tau = \mu \left(\eta - \frac{d_q \mu^{q-1} L^q}{q}\right)$ . Then for  $t \in (0, \min\{1, \frac{1}{\tau}\})$ , the mapping  $T := (I - t\mu F) : C \rightarrow E$  is a contraction with coefficient  $(1 - t\tau)$ .*

**Lemma 2.6** (Zhang and Guo, [21]) *Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $C$  be a nonempty closed convex subset of  $E$ . Suppose  $T : C \rightarrow E$  are  $\lambda$ -strict pseudocontractions such that  $F(T) \neq \emptyset$ . For any  $\alpha \in (0, 1)$ , we define  $T_{\alpha} : C \rightarrow E$  by  $T_{\alpha}x = \alpha x + (1 - \alpha)Tx$ , for each  $x \in C$ . Then, as  $\alpha \in [\mu, 1)$ ,  $\mu \in [\max\{0, 1 - (\frac{\lambda q}{d_q})^{\frac{1}{q-1}}\}, 1)$ ,  $T_{\alpha}$  is a nonexpansive mapping such that  $F(T_{\alpha}) = F(T)$ .*

**Lemma 2.7** (Xu, [19]) *Let  $\{a_n\}$  be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence in  $\mathbb{R}$  such that:

- (i)  $\lim_{n \rightarrow \infty} \gamma_n = 0$  and  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2.8** (Chang et al., [4]) *Let  $E$  be a real  $q$ -uniformly smooth Banach space, then the generalized duality mapping  $J_q : E \rightarrow 2^{E^*}$  is single-valued and uniformly continuous on each bounded subset of  $E$  from the norm topology of  $E$  to the norm topology of  $E^*$ .*

**Lemma 2.9** (Shioji and Takahashi, [13]) *Let  $a$  be a real number and a sequence  $\{a_n\} \in l^\infty$  such that  $\mu_n(a_n) \leq 0$  for all Banach limit  $\mu$  and  $\limsup_{n \rightarrow \infty} (a_{n+1} - a_n) \leq 0$ . Then,  $\limsup_{n \rightarrow \infty} a_n \leq 0$ .*

**Lemma 2.10** (Mitrinović, [11]) *Suppose that  $q > 1$ . Then, for any arbitrary positive real numbers  $x, y$ , the following inequality holds:*

$$xy \leq \frac{1}{q}x^q + \left(\frac{q-1}{q}\right)y^{\frac{q}{q-1}}.$$

**Lemma 2.11** *Let  $E$  be a real  $q$ -uniformly smooth Banach space. Let  $f : E \rightarrow E$  be a contraction mapping with coefficient  $\alpha \in (0, 1)$ . Let  $T : E \rightarrow E$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$  and  $G : E \rightarrow E$  be an  $\eta$ -strongly accretive mapping which is also  $L$ -Lipschitzian. Assume that  $0 < \mu < (\frac{q\eta}{d_q L^q})^{\frac{1}{q-1}}$  and  $0 < \gamma < \frac{\tau}{\alpha}$ , where  $\tau := \mu(\eta - \frac{\mu^{q-1} d_q L^q}{q})$ . Then for each  $t \in (0, \min\{1, \frac{1}{\tau}\})$ , the sequence  $\{x_t\}$  define by*

$$x_t = t\gamma f(x_t) + (I - t\mu G)Tx_t$$

*converges strongly as  $t \rightarrow 0$  to a fixed point  $x^*$  of  $T$  which solves the variational inequality:*

$$\langle (\mu G - \gamma f)x^*, j_q(x^* - x) \rangle \leq 0, \quad \forall x \in F(T). \quad (17)$$

**Proof.** *The definition of  $\{x_t\}$  is well definition.*

*Now, for each  $t \in (0, \min\{1, \frac{1}{\tau}\})$ , define a mapping  $T_t$  on  $C$  by*

$$T_t x = t\gamma f(x) + (I - t\mu G)Tx, \quad \forall x \in C.$$

*Then, by Lemma 2.5, we have*

$$\begin{aligned} \|T_t x - T_t y\| &= \|[t\gamma f(x) + (I - t\mu G)Tx] - [t\gamma f(y) + (I - t\mu G)Ty]\| \\ &\leq \|t\gamma[f(x) - f(y)]\| + \|(I - t\mu G)Tx - (I - t\mu G)Ty\| \\ &\leq t\gamma\|f(x) - f(y)\| + (1 - t\tau)\|Tx - Ty\| \\ &\leq t\gamma\alpha\|x - y\| + (1 - t\tau)\|x - y\| \\ &= [1 - t(\tau - \gamma\alpha)]\|x - y\|, \end{aligned}$$

*which implies that  $T_t$  is a contraction. Hence,  $T_t$  has a unique fixed point, denoted by  $x_t$ , which uniquely solve the fixed point equation:*

$$x_t = t\gamma f(x_t) + (I - t\mu G)Tx_t. \quad (18)$$

We observe that  $\{x_t\}$  is bounded. Indeed, from (18) and Lemma 2.5, we have

$$\begin{aligned} \|x_t - \tilde{x}\| &= \|t\gamma f(x_t) + (I - t\mu G)Tx_t - \tilde{x}\| \\ &= \|t[\gamma f(x_t) - \mu G\tilde{x}] + (I - t\mu G)Tx_t - (I - t\mu G)\tilde{x}\| \\ &\leq \|(I - t\mu G)Tx_t - (I - t\mu G)\tilde{x}\| + t\|\gamma f(x_t) - \mu G\tilde{x}\| \\ &\leq (1 - t\tau)\|x_t - \tilde{x}\| + t\gamma\|f(x_t) - f(\tilde{x})\| + t\|\gamma f(\tilde{x}) - \mu G\tilde{x}\| \\ &\leq (1 - t\tau)\|x_t - \tilde{x}\| + t\gamma\alpha\|x_t - \tilde{x}\| + t\|\gamma f(\tilde{x}) - \mu G\tilde{x}\| \\ &= [1 - t(\tau - \gamma\alpha)]\|x_t - \tilde{x}\| + t\|\gamma f(\tilde{x}) - \mu G\tilde{x}\|. \end{aligned}$$

It follows that

$$\|x_t - \tilde{x}\| \leq \frac{\|\gamma f(\tilde{x}) - \mu G\tilde{x}\|}{\tau - \gamma\alpha}.$$

Hence,  $\{x_t\}$  is bounded. Furthermore  $\{f(x_t)\}$  and  $\{G(Tx_t)\}$  are also bounded.

Also, from (18), we have

$$\|x_t - Tx_t\| = t\|\gamma f(x_t) - \mu G(Tx_t)\| \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (19)$$

Take  $t, t_0 \in (0, \frac{1}{\tau})$ . From (18) and Lemma 2.5, we have

$$\begin{aligned} \|x_t - x_{t_0}\| &= \|[t\gamma f(x_t) + (I - t\mu G)Tx_t] - [t_0\gamma f(x_{t_0}) + (I - t_0\mu G)Tx_{t_0}]\| \\ &= \|(t - t_0)\gamma f(x_t) + t_0\gamma[f(x_t) - f(x_{t_0})] + (t_0 - t)\mu G(Tx_t) \\ &\quad + (I - t_0\mu G)Tx_t - (I - t_0\mu G)Tx_{t_0}\| \\ &\leq (\gamma\|f(x_t)\| + \mu\|G(Tx_t)\|)|t - t_0| + t_0\gamma\|f(x_t) - f(x_{t_0})\| \\ &\quad + \|(I - t_0\mu G)Tx_t - (I - t_0\mu G)Tx_{t_0}\| \\ &\leq (\gamma\|f(x_t)\| + \mu\|G(Tx_t)\|)|t - t_0| + t_0\gamma\alpha\|x_t - x_{t_0}\| + (1 - t_0\tau)\|Tx_t - Tx_{t_0}\| \\ &\leq (\gamma\|f(x_t)\| + \mu\|G(Tx_t)\|)|t - t_0| + [1 - t_0(\tau - \gamma\alpha)]\|x_t - x_{t_0}\|. \end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \leq \frac{\gamma\|f(x_t)\| + \mu\|G(Tx_t)\|}{t_0(\tau - \gamma\alpha)}|t - t_0|.$$

This shows that  $\{x_t\}$  is locally Lipschitzian and hence continuous.

We next show the uniqueness of a solution of the variational inequality (17). Suppose both  $\tilde{x} \in F(T)$  and  $\tilde{y} \in F(T)$  are solutions to (17). From (17), we know that

$$\langle (\mu G - \gamma f)\tilde{x}, j_q(\tilde{x} - \tilde{y}) \rangle \leq 0. \quad (20)$$

and

$$\langle (\mu G - \gamma f)\tilde{y}, j_q(\tilde{y} - \tilde{x}) \rangle \leq 0. \quad (21)$$



Adding up (20) and (21), we have

$$\langle (\mu G - \gamma f)\tilde{x} - (\mu G - \gamma f)\tilde{y}, j_q(\tilde{x} - \tilde{y}) \rangle \leq 0.$$

Observe that

$$\begin{aligned} \frac{d_q \mu^{q-1} L^q}{q} > 0 &\Leftrightarrow \eta - \frac{d_q \mu^{q-1} L^q}{q} < \eta \\ &\Leftrightarrow \mu \left( \eta - \frac{d_q \mu^{q-1} L^q}{q} \right) < \mu \eta \\ &\Leftrightarrow \tau < \mu \eta. \end{aligned}$$

It follows that

$$0 < \gamma \alpha < \tau < \mu \eta.$$

We notice that

$$\begin{aligned} \langle (\mu G - \gamma f)\tilde{x} - (\mu G - \gamma f)\tilde{y}, j_q(\tilde{x} - \tilde{y}) \rangle &= \langle (\mu(G\tilde{x} - G\tilde{y}) - \gamma(f(\tilde{x}) - f(\tilde{y}))), j_q(\tilde{x} - \tilde{y}) \rangle \\ &= \mu \langle G\tilde{x} - G\tilde{y}, j_q(\tilde{x} - \tilde{y}) \rangle - \gamma \langle f(\tilde{x}) - f(\tilde{y}), j_q(\tilde{x} - \tilde{y}) \rangle \\ &\geq \mu \eta \|\tilde{x} - \tilde{y}\|^q - \gamma \|f(\tilde{x}) - f(\tilde{y})\| \|\tilde{x} - \tilde{y}\|^{q-1} \\ &\geq \mu \eta \|\tilde{x} - \tilde{y}\|^q - \gamma \alpha \|\tilde{x} - \tilde{y}\|^q \\ &= (\mu \eta - \gamma \alpha) \|\tilde{x} - \tilde{y}\|^q. \end{aligned}$$

Therefore,  $\tilde{x} = \tilde{y}$  and the uniqueness is proved. Below, we use  $x^* \in F(T)$  to denote the unique solution of the variational inequality (17).

Next, we prove that  $x_t \rightarrow x^*$  as  $t \rightarrow 0$ .

Define a map  $\phi : E \rightarrow \mathbb{R}$  by

$$\phi(x) = \mu_n \|x_n - x\|^q, \quad \forall x \in E,$$

where  $\mu_n$  is a Banach limit for each  $n$ . Then  $\phi$  is continuous, convex, and  $\phi(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ . Since  $E$  is reflexive, there exists  $y^* \in E$  such that  $\phi(y^*) = \min_{u \in E} \phi(u)$ . Hence the set

$$K_{min} := \{x \in E : \phi(x) = \min_{u \in E} \phi(u)\} \neq \emptyset.$$

Therefore, applying Lemma 2.2, we have  $K_{min} \cap F(T) \neq \emptyset$ . Without loss of generality, assume  $x^* = y^* \in K_{min} \cap F(T)$ . Let  $t \in (0, 1)$ . Then, it follows that  $\phi(x^*) \leq \phi(x^* + t(\gamma f - \mu G)x^*)$  and using Lemma 2.4, we obtain that

$$\|x_n - x^* - t(\gamma f - \mu G)x^*\|^q \leq \|x_n - x^*\|^q - qt \langle (\gamma f - \mu G)x^*, j_q(x_n - x^* - t(\gamma f - \mu G)x^*) \rangle.$$

Thus, taking Banach limit over  $n \geq 1$  gives

$$\mu_n \|x_n - x^* - t(\gamma f - \mu G)x^*\|^q \leq \mu_n \|x_n - x^*\|^q - qt \mu_n \langle (\gamma f - \mu G)x^*, j_q(x_n - x^* - t(\gamma f - \mu G)x^*) \rangle.$$

This implies,

$$qt\mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^* - t(\gamma f - \mu G)x^*)\rangle \leq \phi(x^*) - \phi(x^* + t(\gamma f - \mu G)x^*) \leq 0.$$

Therefore

$$\mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^* - t(\gamma f - \mu G)x^*)\rangle \leq 0, \quad \forall n \geq 1.$$

Moreover,

$$\begin{aligned} \mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*)\rangle &= \mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*) - j_q(x_n - x^* - t(\gamma f - \mu G)x^*)\rangle \\ &\quad + \mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^* - t(\gamma f - \mu G)x^*)\rangle \\ &\leq \mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*) - j_q(x_n - x^* - t(\gamma f - \mu G)x^*)\rangle. \end{aligned}$$

By Lemma 2.8, the duality mapping  $J_q$  is norm-to-norm uniformly continuous on bounded subset of  $E$ , we have that

$$\mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*)\rangle \leq 0. \quad (22)$$

Now, using (18) and Lemma 2.5, we have

$$\begin{aligned} \|x_n - x^*\|^q &= t_n\langle\gamma f(x_n) - \mu Gx^*, j_q(x_n - x^*)\rangle + \langle(I - t_n\mu G)(Tx_n - x^*), j_q(x_n - x^*)\rangle \\ &= t_n\langle\gamma f(x_n) - \mu Gx^*, j_q(x_n - x^*)\rangle + \langle(I - t_n\mu G)Tx_n - (I - t_n\mu G)x^*, j_q(x_n - x^*)\rangle \\ &\leq [1 - t_n(\tau - \gamma\alpha)]\|x_n - x^*\|^q + t_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*)\rangle. \end{aligned}$$

So,

$$\|x_n - x^*\|^q \leq \frac{1}{\tau - \gamma\alpha}\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*)\rangle.$$

Again, taking Banach limit, we obtain

$$\mu_n\|x_n - x^*\|^q \leq \frac{1}{\tau - \gamma\alpha}\mu_n\langle(\gamma f - \mu G)x^*, j_q(x_n - x^*)\rangle \leq 0,$$

which implies that  $\mu_n\|x_n - x^*\|^q = 0$ . Hence, there exists a subsequence of  $\{x_n\}$  which will still be denoted by  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} x_n = x^*$ .

We next prove that  $x^*$  solves the variational inequality (17). Since

$$x_t = t\gamma f(x_t) + (I - t\mu G)Tx_t,$$

we can derive that

$$(\mu G - \gamma f)x_t = -\frac{1}{t}(I - T)x_t + \mu(Gx_t - GTx_t)$$

Notice that

$$\begin{aligned} \langle (I - T)x_t - (I - T)z, j_q(x_t - z) \rangle &\geq \|x_t - z\|^q - \|Tx_t - Tz\| \|x_t - z\|^{q-1} \\ &\geq \|x_t - z\|^q - \|x_t - z\|^q \\ &= 0. \end{aligned}$$

It follows that, for all  $z \in F(T)$ ,

$$\begin{aligned} \langle (\mu G - \gamma f)x_t, j_q(x_t - z) \rangle &= -\frac{1}{t} \langle (I - T)x_t - (I - T)z, j_q(x_t - z) \rangle \\ &\quad + \mu \langle Gx_t - G(Tx_t), j_q(x_t - z) \rangle \\ &\leq \mu \langle Gx_t - G(Tx_t), j_q(x_t - z) \rangle \\ &\leq \mu L \|x_t - Tx_t\| \|x_t - z\|^{q-1} \\ &\leq \|x_t - Tx_t\| M, \end{aligned} \tag{23}$$

where  $M$  is an appropriate constant such that  $M = \sup\{\mu L \|x_t - z\|^q\}$ , where  $t \in (0, \min\{1, \frac{1}{\tau}\})$ . Now replacing  $t$  in (23) with  $t_n$  and letting  $n \rightarrow \infty$ , noticing that  $(I - T)x_{t_n} \rightarrow (I - T)x^* = 0$  for  $x^* \in F(T)$ , we obtain  $\langle (\mu G - \gamma f)x^*, x^* - z \rangle \leq 0$ . That is,  $x^* \in F(T)$  is the solution of (17). Hence,  $x^* = \tilde{x}$  by uniqueness. We have shown that each cluster point of  $\{x_t\}$  (at  $t \rightarrow 0$ ) equals  $\tilde{x}$ . Therefore,  $x_t \rightarrow \tilde{x}$  as  $t \rightarrow 0$ . This completes the proof.

### 3 Main Results

**Theorem 3.1** (Synchronal Algorithm)

Let  $E$  be a real  $q$ -uniformly smooth Banach space whose duality map is weakly sequentially continuous and  $C$  be a nonempty, closed and convex subset of  $E$ . Let  $\{T_i\}_{i=1}^\infty : C \rightarrow E$  be a family of  $k$ -strict pseudocontractions for  $k \in (0, 1)$  such that  $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ ,  $f$  be a contraction with coefficient  $\beta \in (0, 1)$  and  $\{\lambda_i\}_{i=1}^\infty$  be a real sequence in  $(0, 1)$  such that  $\sum_{i=1}^\infty \lambda_i = 1$ . Let  $G : C \rightarrow E$  be an  $\eta$ -strongly accretive and  $L$ -Lipschitzian operator with  $L > 0, \eta > 0$ . Assume that  $0 < \mu < (q\eta/d_q L^q)^{1/q-1}, 0 < \gamma < \mu(\eta - d_q \mu^{q-1} L^q/q)/\beta = \tau/\beta$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $(0, 1)$  satisfying the following conditions:

- (K1)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=0}^\infty \alpha_n = \infty;$
- (K2)  $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty;$
- (K3)  $0 < \beta_n < a < 1, \forall n \geq 0.$

Let  $\{x_n\}$  be a sequence defined by the iterative algorithm

$$\begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ T^{\beta_n} = \beta_n I + (1 - \beta_n) \sum_{i=1}^\infty \lambda_i T_i, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T^{\beta_n} x_n, \quad n \geq 0, \end{cases} \tag{24}$$

then  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of  $\{T_i\}_{i=1}^{\infty}$  which solves the variational inequality:

$$\langle (\gamma f - \mu G)x^*, j_q(x - x^*) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \quad (25)$$

**Proof.** Put  $T := \sum_{i=1}^{\infty} \lambda_i T_i$ , then by Lemma 2.1, we conclude that  $T$  is a  $k$ -strict pseudocontraction and  $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$ . We can then rewrite the algorithm (24) as

$$\begin{cases} x_0 \in E \text{ arbitrarily chosen,} \\ T^{\beta_n} = \beta_n I + (1 - \beta_n)T, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T^{\beta_n} x_n, \quad n \geq 0. \end{cases}$$

Furthermore, by using Lemma 2.6, we conclude that  $T^{\beta_n}$  is a nonexpansive mapping and  $F(T^{\beta_n}) = F(T)$ . From the condition (K1), we may assume, without loss of generality, that  $\alpha_n \in (0, \min\{1, \frac{1}{\tau}\})$ . We shall carry out the proof in six steps as follows:

**Step 1.**

We show that  $\{x_n\}$  is bounded.

Take  $p \in \bigcap_{i=1}^{\infty} F(T_i)$ , then the sequence  $\{x_n\}$  satisfies

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|\gamma f(p) - \mu Gp\|}{\tau - \gamma\beta} \right\}, \quad \forall n \geq 0.$$

We prove this by Mathematical induction as follows;

Obviously, it is true for  $n = 0$ . Assume it is true for  $n = k$  for some  $k \in \mathbb{N}$ .

From (24) and Lemma 2.5, we have

$$\begin{aligned} \|x_{k+1} - p\| &= \|\alpha_k \gamma f(x_k) + (I - \alpha_k \mu G)T^{\beta_k} x_k - p\| \\ &= \|\alpha_k [\gamma f(x_k) - \mu Gp] + (I - \alpha_k \mu G)(T^{\beta_k} x_k - p)\| \\ &\leq (1 - \alpha_k \tau) \|x_k - p\| + \alpha_k \|\gamma [f(x_k) - f(p)] + \gamma f(p) - \mu Gp\| \\ &\leq (1 - \alpha_k \tau) \|x_k - p\| + \alpha_k \gamma \beta \|x_k - p\| + \alpha_k \|\gamma f(p) - \mu Gp\| \\ &= [1 - \alpha_k (\tau - \gamma\beta)] \|x_k - p\| + \alpha_k (\tau - \gamma\beta) \frac{\|\gamma f(p) - \mu Gp\|}{\tau - \gamma\beta} \\ &\leq \max \left\{ \|x_k - p\|, \frac{\|\gamma f(p) - \mu Gp\|}{\tau - \gamma\beta} \right\}. \end{aligned}$$

Hence the proved. Thus, the sequence  $\{x_n\}$  is bounded and so are  $\{Tx_n\}$ ,  $\{GT^{\beta_n} x_n\}$  and  $\{f(x_n)\}$ .

**Step 2.**

We show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

Observe that

$$\begin{aligned}
x_{n+2} - x_{n+1} &= [\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}\mu G)T^{\beta_{n+1}}x_{n+1}] \\
&\quad - [\alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)T^{\beta_n}x_n] \\
&= [\alpha_{n+1}\gamma f(x_{n+1}) - \alpha_{n+1}\gamma f(x_n)] + [\alpha_{n+1}\gamma f(x_n) - \alpha_n\gamma f(x_n)] \\
&\quad + [(I - \alpha_{n+1}\mu G)T^{\beta_{n+1}}x_{n+1} - (I - \alpha_{n+1}\mu G)T^{\beta_n}x_{n+1}] \\
&\quad + [\alpha_{n+1}\mu GT^{\beta_n}x_n - \alpha_{n+1}\mu GT^{\beta_n}x_n] \\
&= \alpha_{n+1}\gamma[f(x_{n+1}) - f(x_n)] + [(I - \alpha_{n+1}\mu G)T^{\beta_{n+1}}x_{n+1} \\
&\quad - (I - \alpha_{n+1}\mu G)T^{\beta_n}x_{n+1}] + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\
&\quad + (\alpha_n - \alpha_{n+1})\mu GT^{\beta_n}x_n,
\end{aligned}$$

so that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq \alpha_{n+1}\gamma\beta\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\tau)\|T^{\beta_{n+1}}x_{n+1} - T^{\beta_n}x_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|(\gamma\|f(x_n)\| + \mu\|GT^{\beta_n}x_n\|) \\
&\leq \alpha_{n+1}\gamma\beta\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\tau)\|T^{\beta_{n+1}}x_{n+1} - T^{\beta_n}x_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|M_1,
\end{aligned} \tag{26}$$

where  $M_1$  is an appropriate constant such that  $M_1 \geq \sup_{n \geq 1} \{\gamma\|f(x_n)\| + \mu\|GT^{\beta_n}x_n\|\}$ .  
On the other hand, we note that

$$\begin{aligned}
\|T^{\beta_{n+1}}x_{n+1} - T^{\beta_n}x_n\| &\leq \|T^{\beta_{n+1}}x_{n+1} - T^{\beta_{n+1}}x_n\| + \|T^{\beta_{n+1}}x_n - T^{\beta_n}x_n\| \\
&\leq \|x_{n+1} - x_n\| + \|[\beta_{n+1}x_{n+1} + (1 - \beta_{n+1})Tx_{n+1}] \\
&\quad - [\beta_{n+1}x_n + (1 - \beta_{n+1})Tx_n]\| \\
&= \|x_n - x_n\| + \|\beta_{n+1}(x_n - Tx_n) - \beta_n(x_n - Tx_n)\| \\
&\leq \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|M_2,
\end{aligned} \tag{27}$$

where  $M_2$  is an appropriate constant such that  $M_2 \geq \sup_{n \geq 1} \{\|x_n - Tx_n\|\}$ .

Now, substituting (27) into (26) yields

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq \alpha_{n+1}\gamma\beta\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\tau)\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|M_1 \\
&\quad + |\beta_{n+1} - \beta_n|M_2 \\
&\leq [1 - \alpha_{n+1}(\tau - \gamma\beta)]\|x_{n+1} - x_n\| + (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|)M_3,
\end{aligned}$$

where  $M_3$  is an appropriate constant such that  $M_3 \geq \max\{M_1, M_2\}$ .

By Lemma 2.7 and the conditions (K1), (K2), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{28}$$

**Step 3.**

We show that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .

From (24) and condition (K1), we have

$$\begin{aligned}\|x_{n+1} - T^{\beta_n}x_n\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)T^{\beta_n}x_n - T^{\beta_n}x_n\| \\ &\leq \alpha_n\|\gamma f(x_n) + \mu GT^{\beta_n}x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}\quad (29)$$

On the other hand,

$$\begin{aligned}\|x_{n+1} - T^{\beta_n}x_n\| &= \|x_{n+1} - [\beta_n x_n + (1 - \beta_n)Tx_n]\| \\ &= \|x_{n+1} - x_n + (1 - \beta_n)(x_n - Tx_n)\| \\ &\leq \|x_{n+1} - x_n\| + (1 - \beta_n)\|x_n - Tx_n\|,\end{aligned}$$

which implies, by condition (K3), that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \frac{1}{1 - \beta_n}(\|x_{n+1} - x_n\| + \|x_{n+1} - T^{\beta_n}x_n\|) \\ &\leq \frac{1}{1 - a}(\|x_{n+1} - x_n\| + \|x_{n+1} - T^{\beta_n}x_n\|).\end{aligned}$$

Hence, from (28) and (29), we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (30)$$

**Step 4.**

We show that  $\omega_\omega(x_n) \subset F(T)$ .

From the boundedness of  $\{x_n\}$ , without loss of generality, we may assume that  $x_n \rightharpoonup y$ . Hence, by Lemma 2.3 and (30), we obtain  $Ty = y$ . So, we have

$$\omega_\omega(x_n) \subset F(T). \quad (31)$$

**Step 5.**

We show that  $\limsup_{n \rightarrow \infty} \langle (\gamma f - \mu G)x^*, j_q(x_n - x^*) \rangle \leq 0$ ,

where  $x^*$  is obtained in Lemma 2.11. Put  $a_n := \langle (\gamma f - \mu G)x^*, j_q(x_n - x^*) \rangle$ . Then, by (22), we have  $\mu_n(a_n) \leq 0$  for any Banach limit  $\mu$ . Furthermore, by (28),  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , we therefore conclude that

$$\begin{aligned}\limsup_{n \rightarrow \infty} (a_{n+1} - a_n) &= \limsup_{n \rightarrow \infty} (\langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle - \langle (\gamma f - \mu G)x^*, j_q(x_n - x^*) \rangle) \\ &= \limsup_{n \rightarrow \infty} \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) - j_q(x_n - x^*) \rangle = 0.\end{aligned}$$

Hence, by Lemma 2.9, we have  $\limsup_{n \rightarrow \infty} a_n \leq 0$ , that is,

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - \mu G)x^*, j_q(x_n - x^*) \rangle \leq 0. \quad (32)$$

**Step 6.**

We show that  $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ .

Using (24), Lemmas 2.5 and 2.10, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \langle x_{n+1} - x^*, j_q(x_{n+1} - x^*) \rangle \\
&= \langle \alpha_n[\gamma f(x_n) - \mu Gx^*] + (I - \alpha_n \mu G)(T^{\beta_n} x_n - x^*), j_q(x_{n+1} - x^*) \rangle \\
&= \alpha_n \langle \gamma f(x_n) - \gamma f(x^*), j_q(x_{n+1} - x^*) \rangle + \alpha_n \langle \gamma f(x^*) - \mu Gx^*, j_q(x_{n+1} - x^*) \rangle \\
&\quad + \langle (I - \alpha_n \mu G)(T^{\beta_n} x_n - x^*), j_q(x_{n+1} - x^*) \rangle \\
&\leq \alpha_n \gamma \|f(x_n) - f(x^*)\| \|x_{n+1} - x^*\|^{q-1} + \alpha_n \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle \\
&\quad + \|(I - \alpha_n \mu G)(T^{\beta_n} x_n - x^*)\| \|x_{n+1} - x^*\|^{q-1} \\
&\leq (1 - \alpha_n \tau) \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \alpha_n \gamma \beta \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} \\
&\quad + \alpha_n \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle \\
&\leq [1 - \alpha_n(\tau - \gamma\beta)] \left[ \frac{1}{q} \|x_n - x^*\|^q + \left(\frac{q-1}{q}\right) \|x_{n+1} - x^*\|^q \right] \\
&\quad + \alpha_n \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &\leq \frac{1 - \alpha_n(\tau - \gamma\beta)}{1 + \alpha_n(q-1)(\tau - \gamma\beta)} \|x_n - x^*\|^q \\
&\quad + \frac{q\alpha_n}{1 + \alpha_n(q-1)(\tau - \gamma\beta)} \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle \\
&\leq [1 - \alpha_n(\tau - \gamma\beta)] \|x_n - x^*\|^q \\
&\quad + \frac{q\alpha_n}{1 + \alpha_n(q-1)(\tau - \gamma\beta)} \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle \\
&\leq (1 - \gamma_n) \|x_n - x^*\|^q + \delta_n,
\end{aligned}$$

where  $\gamma_n := \alpha_n(\tau - \gamma\beta)$  and  $\delta_n := \frac{q\alpha_n}{1 + \alpha_n(q-1)(\tau - \gamma\beta)} \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle$ . From (K1),

$\lim_{n \rightarrow \infty} \gamma_n = 0$ ,  $\sum_{n=0}^{\infty} \gamma_n = \infty$ . Now  $\frac{\delta_n}{\gamma_n} = \frac{q}{[1 + \alpha_n(q-1)(\tau - \gamma\beta)](\tau - \gamma\beta)} \langle (\gamma f - \mu G)x^*, j_q(x_{n+1} - x^*) \rangle$ . So,  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ . Hence, by Lemma 2.7, we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0.$$

This completes the proof.

## 4 Conclusion

The following Corollaries are consequences of Theorem 3.1.

**Corollary 4.1** Let  $E$  be a real  $q$ -uniformly smooth Banach space whose duality map is weakly sequentially continuous and  $C$  be a nonempty, closed and convex subset of  $E$ . Let  $\{T_i\}_{i=1}^N : C \rightarrow E$  be a family of  $k_i$ -strict pseudocontractions for  $k_i \in (0, 1)$ , ( $i = 1, 2, \dots, N$ ), such that  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$  and  $k = \min\{k_i : 1 \leq i \leq N\}$ ,  $f$  be a contraction with coefficient  $\beta \in (0, 1)$  and  $\{\lambda_i\}_{i=1}^N$  be a real sequence in  $(0, 1)$  such that  $\sum_{i=1}^N \lambda_i = 1$ . Let  $G : C \rightarrow E$  be an  $\eta$ -strongly accretive and  $L$ -Lipschitzian operator with  $L > 0$ ,  $\eta > 0$ . Assume that  $0 < \mu < (q\eta/d_q L^q)^{1/q-1}$ ,  $0 < \gamma < \mu(\eta - d_q \mu^{q-1} L^q/q)/\beta = \tau/\beta$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $(0, 1)$  satisfying the following conditions (K1)–(K3). Let  $\{x_n\}$  be a sequence defined by the iterative algorithm

$$\begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ T^{\beta_n} = \beta_n I + (1 - \beta_n) \sum_{i=1}^N \lambda_i T_i, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T^{\beta_n} x_n, \quad n \geq 0, \end{cases} \quad (33)$$

then  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of  $\{T_i\}_{i=1}^N$  which solves the variational inequality:

$$\langle (\gamma f - \mu G)x^*, j_q(x - x^*) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^N F(T_i). \quad (34)$$

**Corollary 4.2** Let  $E$  be a real 2-uniformly smooth Banach space whose duality map is weakly sequentially continuous and  $C$  be a nonempty, closed and convex subset of  $E$ . Let  $\{T_i\}_{i=1}^\infty : C \rightarrow E$  be a family of  $k$ -strict pseudocontractions for  $k \in (0, 1)$ , such that  $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ ,  $f$  be a contraction map with coefficient  $\beta \in (0, 1)$  and  $\{\lambda_i\}_{i=1}^\infty$  be a real sequence such that  $\sum_{i=1}^\infty \lambda_i = 1$ . Let  $G : C \rightarrow E$  be an  $\eta$ -strongly accretive and  $L$ -Lipschitzian operator with  $L > 0$ ,  $\eta > 0$ . Assume that  $0 < \mu < 2\eta/d_2 L^2$ ,  $0 < \gamma < \mu(\eta - d_2 \mu L^2/2)/\beta = \tau/\beta$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $(0, 1)$  satisfying the conditions (K1)–(K3).

Let  $\{x_n\}$  be a sequence defined by the algorithm (24), then  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of  $\{T_i\}_{i=1}^\infty$  which solves the variational inequality:

$$\langle (\gamma f - \mu G)x^*, j(x - x^*) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^\infty F(T_i). \quad (35)$$

**Corollary 4.3** Let  $E = L_p$  (or  $l_p$ ) space, ( $1 < p < \infty$ ). Let  $\{T_i\}_{i=1}^\infty : E \rightarrow E$  be  $k_i$ -strict pseudocontractions for  $k_i \in (0, 1)$  such that  $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ ,  $f$  be a contraction map with coefficient  $\beta \in (0, 1)$  and  $\lambda_i$  be positive constants such that  $\sum_{i=1}^\infty \lambda_i = 1$ . Let  $G : E \rightarrow E$  be an  $\eta$ -strongly accretive and  $L$ -Lipschitzian operator with  $L > 0$ ,  $\eta > 0$ . Assume that  $0 < \mu < 2\eta/d_2 L^2$ ,  $0 < \gamma < \mu(\eta - d_2 \mu L^2/2)/\beta = \tau/\beta$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $(0, 1)$  satisfying the conditions (K1)–(K3).

Let  $\{x_n\}$  be a sequence defined by the composite process (24), then  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of  $\{T_i\}_{i=1}^\infty$  which solves the variational inequality (35).



**Corollary 4.4** (Tian and Di, [18]) *Let  $E = H$  be a real Hilbert space. Let  $\{x_n\}$  be a sequence generated by (15). Assume that  $\{\alpha_n\}$  are  $\{\beta_n\}$  are sequences in  $(0, 1)$  satisfying the conditions (K1)–(K3), then  $\{x_n\}$  converges strongly to a common fixed point  $x^*$  of  $\{T_i\}_{i=1}^N$  which solves the variational inequality (16).*

**Corollary 4.5** (Tian, [17]) *Let  $E = H$  be a real Hilbert space. Let  $\{x_n\}$  be a sequence generated by (33). Assume that  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying (K1) and (K2), then  $\{x_n\}$  converges strongly to a common fixed point of  $T$  which solves the variational inequality (14).*

**Corollary 4.6** (Marino and Xu, [10]) *Let  $E = H$  be a real Hilbert space. Let  $\{x_n\}$  be a sequence generated by (11). Assume that  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying (K1) and (K2), then  $\{x_n\}$  converges strongly to a common fixed point of  $T$  which solves the variational inequality (12).*

**Corollary 4.7** (Yamada, [20]) *Let  $E = H$  be a real Hilbert space. Let  $\{x_n\}$  be a sequence generated by (9). Assume that  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying (K1) and (K2), then  $\{x_n\}$  converges strongly to a common fixed point of  $T$  which solves the variational inequality (10).*

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