P.1 Real Numbers & The Real Line :

Real numbers are ~~the~~ numbers that can be expressed as decimals ,

Ex: $-\frac{3}{4} = -0.7500\ldots$ $\sqrt{2} = 1.4142\ldots$

$4 = 4.000\ldots$ $\pi = 3.14159$

$\frac{1}{3} = 0.333\ldots$

The order properties of real numbers :

If a, b and c are real numbers then

1.) $a < b \Rightarrow a + c < b + c$

2.) $a < b \Rightarrow a - c < b - c$

3.) $a < b$ and $c > 0 \Rightarrow ac < bc$

4.) $a < b$ and $c < 0 \Rightarrow ac > bc$

5.) $a > 0 \Rightarrow \frac{1}{a} > 0$

6.) $0 < a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

The Natural Numbers (positive integers) : $1, 2, 3, 4 \ldots$

The Integers : $0, \pm 1, \mp 2, \ldots$

The Rational Numbers : The numbers that can be expressed in the form of a fraction $\frac{m}{n}$ m, n : integers, $n \neq 0$

The rational numbers are those real numbers with decimal expansions that are either:

(a.) terminating (ending with an infinite string of zeros)

Ex: $\frac{3}{4} = 0.75000\ldots$

(b.) repeating (ending with a string of digits that repeat: over and over

Ex: $\frac{23}{11} = 2.0909\ldots = 2.\overline{09}$

Real numbers that are not rational are called irrational numbers

Intervals:

A subset of the real line is called an interval if it contains at least two numbers and also contains all real numbers between any two of its elements.

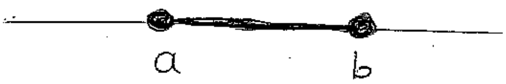
Ex: ~~$x < 3$~~ The set of all real numbers x s.t $x < 3$

Let a and b are real numbers and $a < b$

(1.) Open Interval (a, b)

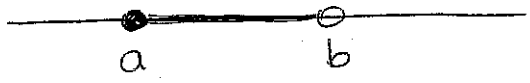
, $a < x < b$

(2.) The closed interval $[a, b]$



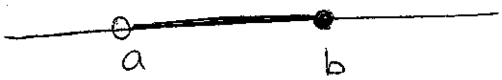
, $a \leq x \leq b$

3) The Half-open interval $[a, b)$



, $a \leq x < b$

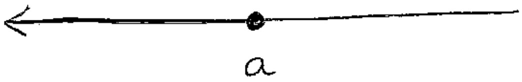
4.) The Half-open interval $(a, b]$



, $a < x \leq b$

Infinite Intervals:

the interval (a, ∞)



the interval $(-\infty, a]$

Ex1 Solve the following inequalities

a.) $5X-3 \leq 7-3X$

$$8X \leq 10$$

$$X \leq \frac{10}{8}$$

sol set $= (-\infty, \frac{10}{8})$

b.) $\frac{x}{2} \leq 1+3x$

$x \leq 2+6x$

$-2 \leq 5x$

$-\frac{2}{5} \leq x$ Sol set = $[-\frac{2}{5}, \infty)$

c.) $\frac{2}{x-1} \geq 5$ Do not make cross multiplication use sign table.

$\frac{2}{x-1} - 5 \geq 0$

$\frac{2-5(x-1)}{x-1} \geq 0$

$\frac{2-5x+5}{x-1} \geq 0$

$\frac{-5x+7}{x-1} \geq 0$

$-5x+7=0$ $x-1=0$

$x = \frac{7}{5} = 1\frac{2}{5}$ $x=1$

x		1		$\frac{7}{5}$	
$x-1$	$-$	0	$+$		$+$
$-5x+7$	$+$		$+$	0	$-$
$\frac{-5x+7}{x-1}$	$-$		$+$		$-$

Sol set = $(1, \frac{7}{5}]$

Ex 2 Solve the following system of inequalities

(a.) $3 \le 2x+1 \le 5$

$3 \le 2x+1$ and ($\cap$) $2x+1 \le 5$

$2 \le 2x$ $\qquad$ $2x \le 4$

$1 \le x$ $\qquad$ $x \le 2$

sol set = $[1,2]$

(b.) $3x-1 < 5x+3 \le 2x+15$

$3x-1 < 5x+3$ (and $\cap$) $5x+3 \le 2x+15$

$-4 < 2x$ $\qquad$ $3x \le 12$

$-2 < x$ $\qquad$ $x \le 4$

soln set : $(-2,4]$

Ex 3 Solve the following quadradic inequalities:

a.) $x^2-x \le 2$

$x^2-x-2 \le 0$

$\Delta = (-1)^2-4(1)(-2) = 1+8 = 9 > 0 \rightarrow$ two real roots

$$x_{1,2} = \frac{1 \mp \sqrt{9}}{2} = \frac{1 \mp 3}{2} \quad \begin{cases} x_1 = 2 \\ x_2 = -1 \end{cases}$$

		-1		2	
$x^2 - x - 2$	$+$	●	$-$	●	$+$

soln set $= [-1, 2]$

(b.) $x^2 + 2x > +15$

$x^2 + 2x - 15 > 0$

$\Delta = 4 - 4(1)(-15) = 64$ → two real roots

$$x_{1,2} = \frac{-2 \mp \sqrt{64}}{2(1)} = \frac{-2 \mp 8}{2} = \begin{cases} x_1 = -5 \\ x_2 = 3 \end{cases}$$

x		-5		3	
$x^2 + 2x - 15$	$+$	o	$-$	o	$+$

sol set $= (-\infty, -5) \cup (3, +\infty)$

Ex 4 Solve, $\dfrac{3}{x-1} < \dfrac{2}{x+1}$

$$\frac{3}{x-1} - \frac{2}{x+1} < 0$$

$$\frac{3(x+1) - 2(x-1)}{(x-1)(x+1)} < 0$$

$$\frac{3x+3-2x+2}{x^2-1} < 0.$$

$$\frac{x+5}{x^2-1} < 0.$$

		-5		-1		1	
$x+5$	$-$	0	$+$		$+$		$+$
x^2-1	$+$		$+$	0	$-$	0	$+$
$\dfrac{x+5}{x^2-1}$	$-$		$+$		$-$		$+$

sol set $= (-\infty, -5) \cup (-1, 1)$

Burda

The Absolute Value:

The absolute value or magnitude of x is,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Ex: $|-2| = 2$, $|4| = 4$, $|0| = 0$

Properties of Absolute Values

1.) $|-a| = |a|$

2.) $|ab| = |a|\,|b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

3.) $|a \pm b| \leq |a| + |b|$ Triangle Inequality.

Remark: $\sqrt{a^2} = |a|$

if we know that $a \geq 0$ then $\sqrt{a^2} = a$.

Equations and Inequalities Involving Absolute Values:

If D is a positive number,

$\|x\| = D$	$\Leftrightarrow$	either $x = D$ or $x = -D$
$\|x\| < D$	$\Leftrightarrow$	$-D < x < D$
$\|x\| \leq D$	$\Leftrightarrow$	$-D \leq x \leq D$
$\|x\| > D$	$\Leftrightarrow$	either $x < -D$ or $x > D$

Ex: Solve,

(a.) $|2x+5| = 4$

$\Leftrightarrow$ either $2x+5 = 4$ or $2x+5 = -4$

$2x = -1$ $\qquad$ $2x = -9$

$x = -\frac{1}{2}$ $\qquad$ $x = -\frac{9}{2}$

(b.) $|3x-7| < 2$

$\Leftrightarrow$ $-2 < 3x-7 < 2$

I. way $\quad -2 < 3x-7$ and $3x-7 < 2$

$5 < 3x$ $\qquad$ $3x < 9$

$\frac{5}{3} < x$ $\qquad$ $x < 3$

II. way

$5 < 3x < 9$

$\frac{5}{3} < x < 3$

soln: $\left(\frac{5}{3}, 3\right)$

(vakit kalırsa).

(c.) $\left|2 - \frac{x}{2}\right| < \frac{1}{2}$

$-\frac{1}{2} < 2 - \frac{x}{2} < \frac{1}{2}$

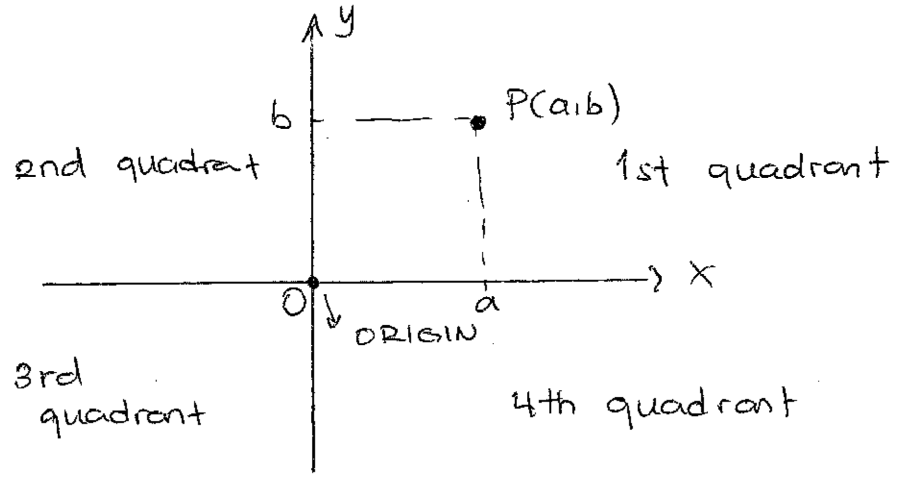
$-\frac{1}{2} - 2 < -\frac{x}{2} < \frac{1}{2} - 2$

$-\frac{5}{2} < \frac{-x}{2} < -\frac{3}{2}$

$-5 < -x < -3$

$5 > x > 3$

Cartesian Coordinates In the Plane :

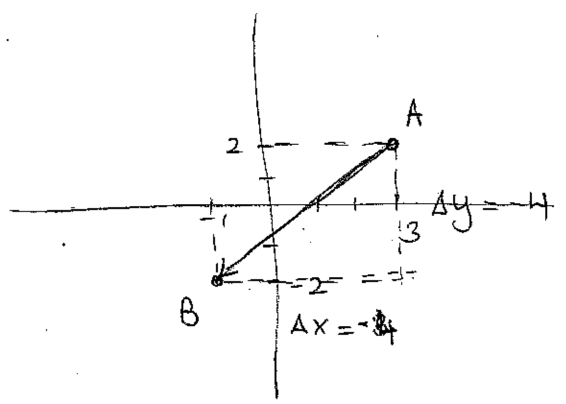


Increments and Distances :

An increment in a variable is the net change in the value of the variable.

If x changes from x_1 to x_2 then the increment in x is $\Delta x = x_2 - x_1$

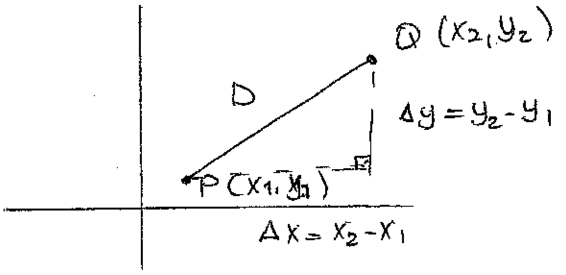
Ex: Find the increments in the coordinates of a particle that moves from $A(3,2)$ to $B(-1,-2)$.



$\Delta x = x_2 - x_1 = -1 - 3 = -4$

$\Delta y = y_2 - y_1 = -2 - 2 = -4$

Distance Formula for Points in the plane :



The distance D between, $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\boxed{D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Ex: Find the distance between $A\,(3,2)$ and $B\,(-1,-2)$.

$$D = \sqrt{(-1-3)^2 + (-2-2)^2} = \sqrt{16+16} = \sqrt{32} \;//.$$

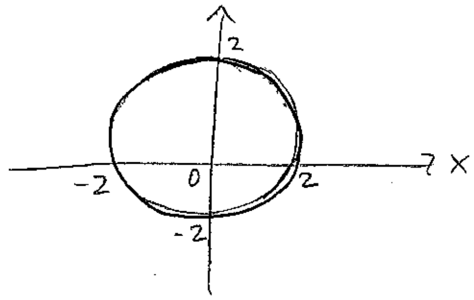
Ex: Find the distance from the origin $O\,(0,0)$ to a point $P(x,y)$.

$$D = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Graphs :

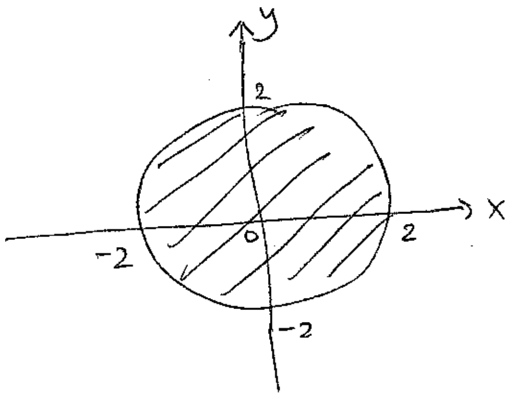
The graph of an eqn (inequality) involving the variables x and y is the set of all points $P(x,y)$ whose coordinates satisfy the eqn (or inequality).

Ex: $x^2+y^2=4$.



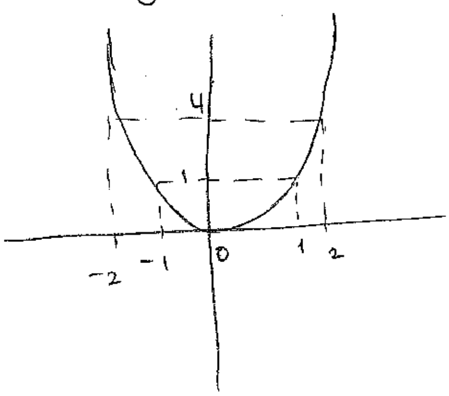
all points $P(x,y)$ whose distance from the origin is 2.

Ex: $x^2+y^2 \le 4$



all points whose distance is less than 2.

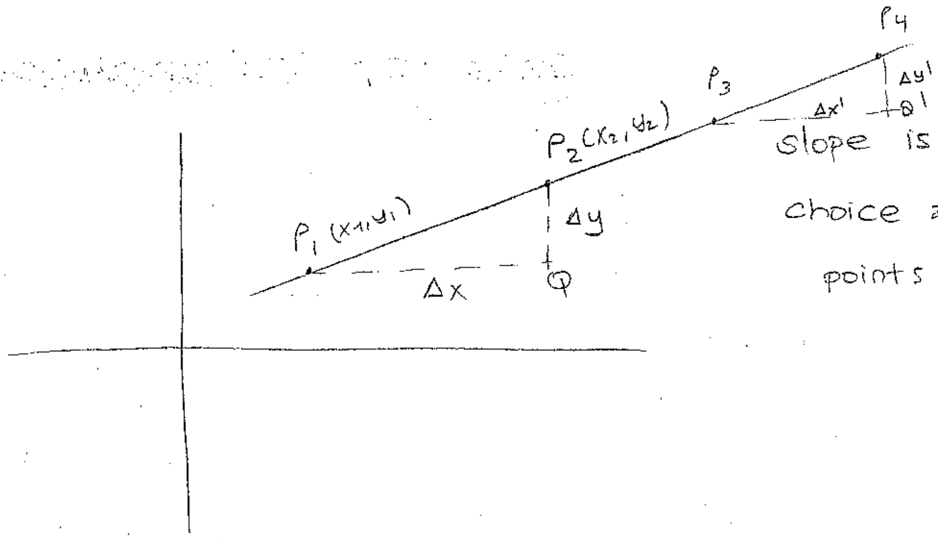
Ex: $y = x^2$



Straight Lines :

Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ determine a unique straight line passing through them.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$ is called slope of the line.



slope is same for ∀ choice of two distinct points P_1 and P_2 on line

Ex: Find the slope of the line passing through A (2, 3) B(−1, 4)

$$m = \frac{\Delta y}{\Delta x} = \frac{4-3}{-1-2} = -\frac{1}{3}$$

* The slope tells us the direction and steepness of a line.

* Line with positive slope rises uphill to the right.
" " negative slope falls downhill " "

* The greater the abs value of slope, the steeper the rise or fall.

* Slope of a vertical line is undefined

* Two lines (L_1 and L_2) are <u>parallel</u> $\Leftrightarrow$ they have same slope.
$\boxed{m_1 = m_2}$.

* If two lines (L_1 and L_2) are <u>perpendicular</u> then,

$\boxed{m_1 \cdot m_2 = -1}$ ie/ $m_1 = -\frac{1}{m_2}$, $m_2 = -\frac{1}{m_1}$

Equations of Lines :

$x = a$ eqn of the vertical line meeting x-axis at a
$y = b$ " " horizontal " " y-axis " b

To write an eqn of line we need

- its slope
- one point $P_1(x_1, y_1)$ on it.

Let $P(x, y)$ another point on the line then,

$$m = \frac{y - y_1}{x - x_1}$$

$\Rightarrow$ $m(x - x_1) = y - y_1$

$\Rightarrow$ $\boxed{y = m(x - x_1) + y_1}$ point-slope eqn of line which passes through (x_1, y_1) and has slope m

Ex: Find an eqn of the line of slope $\frac{1}{2}$ through the point $(-2,2)$.

$$y - y_1 = m\,(x - x_1)$$

$$y = m\,(x - x_1) + y_1$$

$$y = \frac{1}{2}(x+2) + 2$$

$$\boxed{y = \frac{1}{2}x + 3}$$

Ex: Find an eqn of the line through the points $(4,1)$ and $(-2,3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-2-4} = \frac{2}{-6} = -\frac{1}{3}$$

$$y - y_1 = m\,(x - x_1)$$

$$y = m\,(x - x_1) + y_1$$

$$y = -\frac{1}{3}(x-4) + 1 \Rightarrow \boxed{y = -\frac{1}{3}x + \frac{7}{3}}$$

y-intercept: the y-coordinate of the point where line intersects the y-axis.

x-intercept: the x-coordinate of the point where it crosses the x-axis.

A line with slope m and y-int b. (0, b)

$y = m(x-0)+b \Rightarrow \boxed{y = mx+b}$

slope-y-int eq of line with slope m and y-int b.

A line with slope m and x-int a., passes through. (a, 0)

$\boxed{y = m(x-a)}$

slope x-intercept eqn of line with slope m and x-int a.

Ex: Find the slope and two intercepts of the line with eqn

$3x + 4y = 12$.

Soln: $4y = 12 - 3x$

$y = -\frac{3}{4}x + 3 \Rightarrow$ Slope $= -\frac{3}{4}$ //

x-int: let $y=0$ $3x = 12 \Rightarrow x = 4$ $\therefore (4,0)$ x-int. //

y-int: Let $x=0$ $4y = 12 \Rightarrow y = 3$ $\therefore (0,3)$ y-int

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The eqn $Ax + By = C$ where A & B are not both ze'.

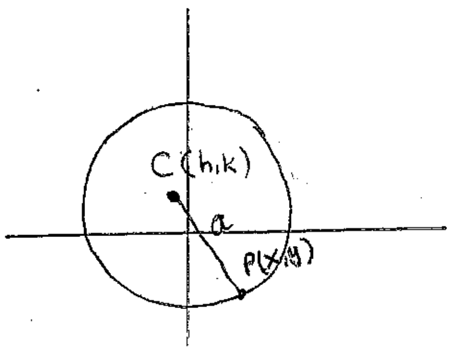
is called the General Linear Eqn in x and y

Its graph is a straight line

P.3 Graphs of Quadratic Equations :

Circles and Disks :

The circle having centre C and radius a, is the set of all points in the plane that are at distance a from the point C



distance from P(x,y) to C(h,k) $= \sqrt{(x-h)^2 + (y-k)^2} = c$

Eqn of circle

$$\boxed{(x-h)^2 + (y-k)^2 = a^2}$$

with center (h,k) and radius $a > 0$

$$\boxed{x^2 + y^2 = a^2}$$

circle with center (0,0)

Ex: write an eqn for the circle with centre C(3,-4) and radius $r = 5$

$$(x-3)^2 + (y+4)^2 = 25$$

Ex: Find the radius and centre of the circle having eqn

$$(x+2)^2 + (y+1)^2 = 9$$

Centre $= (-2, 1)$ radius $= 3$.

A quadratic eqn of the form

$$\boxed{x^2+y^2+2ax+2by=c}$$

must represent a circle, a single point or no points at al

To identify graph;

$$\boxed{(x+a)^2+(y+b)^2=c+a^2+b^2}$$

1. if $c+a^2+b^2>0$ $\Rightarrow$ graph is a circle with centre $(-a,$
and $r=\sqrt{c+a^2+b^2}$

2) if $c+a^2+b^2=0$ $\Rightarrow$ graph is a single point $(-a,-$

3. if $c+a^2+b^2<0$ $\Rightarrow$ no points lie on the graph.

Ex: Find the centre and radius of the circle

$$x^2+y^2-2x+4y=4.$$

Soln:
$$(x-1)^2+(y+2)^2=4+1+4$$

$$(x-1)^2+(y+2)^2=9$$

centre $=(1,-2)$ radius $=3$.

Defn :

* The set of all points inside a circle is called interior of the circle. The interior of a circle is sometimes called an open disk

Defn: The set of all points outside the circle is called the exterior of the circle.

Defn: The interior of a circle together with the circle itself is called a closed disk or simply disk

$$\boxed{(x-h)^2 + (y-k)^2 \le a^2}$$ represents the disk of radius $|a|$ centered at $(h,k$

Ex: Identify the graphs of

(a.) $x^2 + 2x + y^2 \le 8$

(b.) $x^2 + 2x + y^2 < 8$

(c) $x^2 + 2x + y^2 > 8$

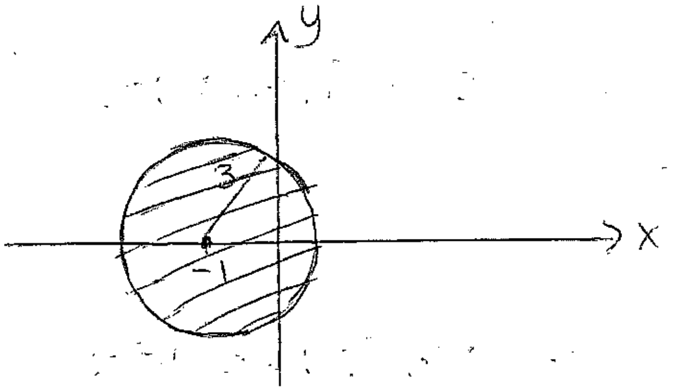
Soln: Complete square in eqn. $x^2 + 2x + y^2 = 8$.

$(x+1)^2 + (y+0)^2 = 8+1$

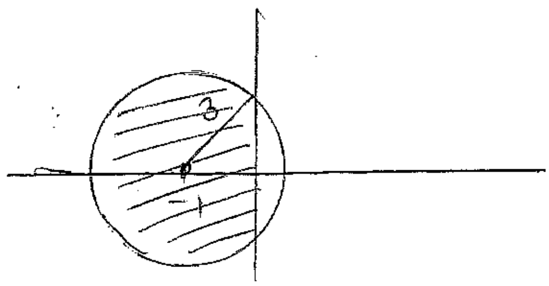
$(x+1)^2 + (y+0)^2 = 9$. $\rightarrow$ eqn of circle with centre $(-1,0)$ and radius $= 3$.

thus,

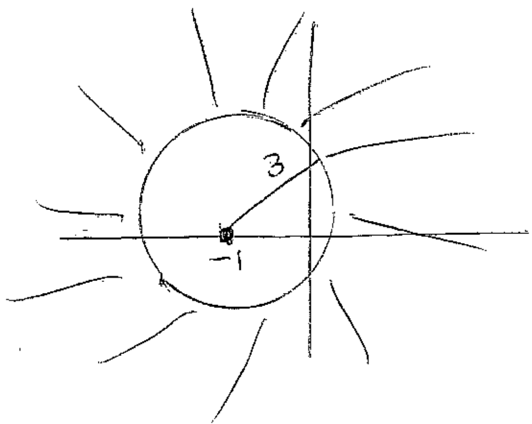
(a.) $x^2 + 2x + y^2 \leq 8$ → represents the closed disk with radius = 3 centre = (-1,0)



(b.) $x^2 + 2x + y^2 < 8$ → represents the interior of the circle.



(c.) $x^2 + 2x + y^2 > 8$ → represents the exterior of the circle.



Eqns of Parabolas :

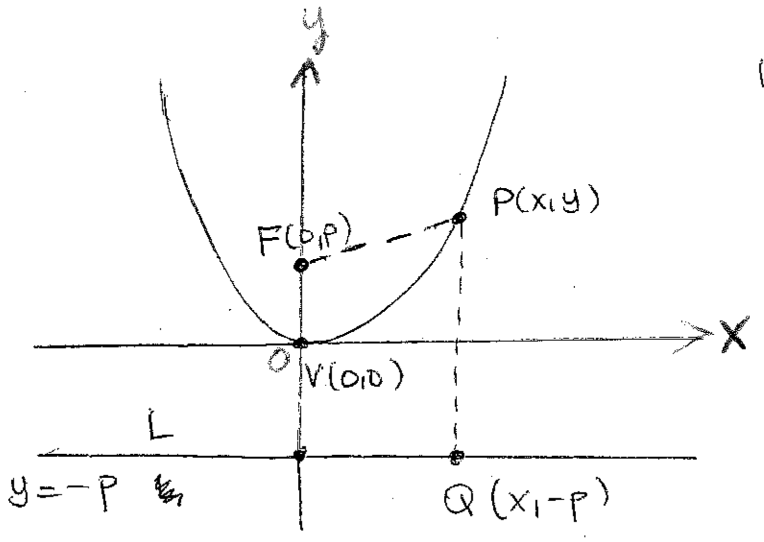
Defn: A parabola is a plane curve whose points are equidistant from a fixed point F and a fixed straight line L that does not pass through F.

The point F is the focus of the parabola the line L is the parabola's directrix

The line through F perpendicular to L is the parabola axis.

The point V where the axis meets the parabola is the parabola's vertex

Ex: Find an eqn of the parabola having the point $F(0,p)$ as focus and the line L with eqn $y=-p$ as directrix.



Let $P(x,y)$ any point on the parabola

$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2+y^2-2py+p^2}$

$PQ = \sqrt{(x-x)^2 + (y+p)^2} = \sqrt{y^2+2py+p^2}$

$PF = PQ$ so, $x^2+y^2-2py+p^2 = y^2+2py+p^2$

$$\boxed{\begin{array}{c} x^2 = 4py \\ y = \dfrac{x^2}{4p} \end{array}}$$ standard forms.

<u>Note:</u> If $p<0$ the focus $(0,p)$ will lie below the origin and the directrix $y=-p$ will lie above the origin. The parabola will open downward.

<u>Ex:</u> Find ~~the~~ eqn for the parabola with focus $(0,4)$ and directrix $y=-4$.

<u>Soln</u> $y = \dfrac{x^2}{4p}$ $\quad p=4 \Rightarrow y = \dfrac{x^2}{16}$

<u>or</u> $16y = x^2$

<u>Ex:</u> Find the focus and directrix of the parabola

$y = \dfrac{x^2}{2}$

<u>Soln:</u> $4p=2 \Rightarrow p=\frac{1}{2}$ $\quad$ Focus $= (0, \frac{1}{2})$ $\quad$ directrix: $\boxed{y = -\frac{1}{2}}$

Interchanging the role of x and y in standard eqn.

$y^2 = 4px \Rightarrow \boxed{x = \frac{y^2}{4p}}$ represents parabola with focus at $(p,0)$

& vertical directrix $x = -p$

The axis is x-axis.

Note: Parabolas are used as reflectors of light and radio wavex.

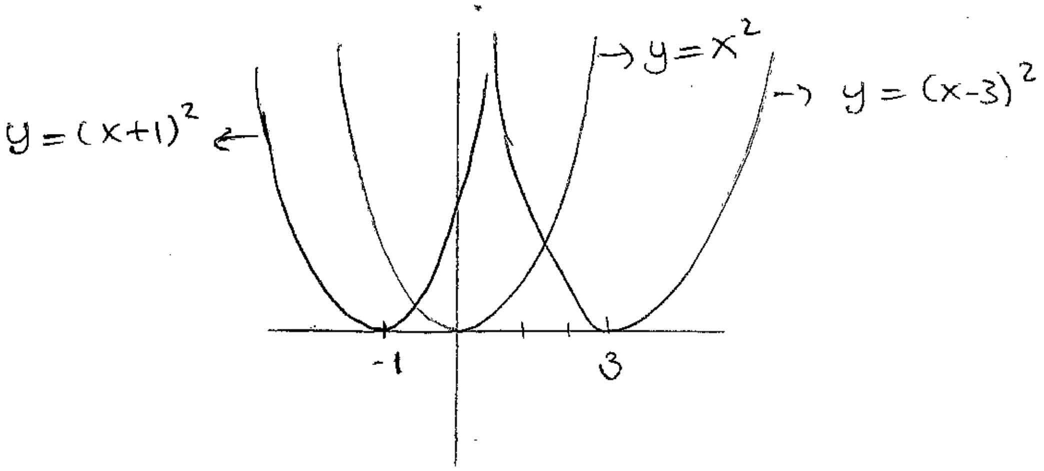
Shifting a Graph:

* To shift a graph c units to the right, replace x in its eqn or ineq. with $x - c$. (if $c < 0$, the shift will be to the left)

* To shift a graph c units upward, replace y in its eqn or ineq. with $y - c$. (if $c < 0$, shift will be downward).

Ex: The graph of $y = (x-3)^2$ is the parabola $y=$ shifted 3 units to the right.

The graph of $y = (x+1)^2$ is the parabola $y = x^2$ shifted 1 unit to the left.

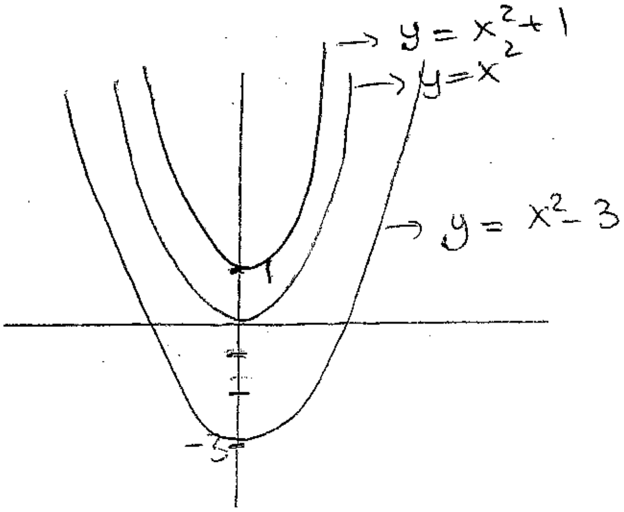


<u>Ex:</u> The graph of $y=x^2+1$, ie/ $y-1=x^2$

is the parabola, $y=x^2$ shifted upward 1 unit.

The graph of $y=x^2-3$, ie/ $y+3=x^2$

is the parabola, $y=x^2$ shifted downward 3 units.



<u>Ex:</u> (2). $(x-h)^2+(y-k)^2=a^2$ centre $=(h,k)$ radius $=a$

(1) $x^2+y^2=a^2$ centre $=(0,0)$, radius $=a$

(2) is obtained from (1) by shifting h units to the right and k units upward.

Bwar

The graph of $\boxed{y = ax^2+bx+C}$ is parabolo

Its axis is parallel to y-axis.

if $a>0 \Rightarrow$ parabola opens upword

if $a<0 \Rightarrow$ " "downward

We can complete square so,

$$\boxed{y = a(x-h)^2+k \quad \Rightarrow \text{ vertex} = (h,k)}$$

Ex: Describe the graph of $y = x^2-4x+3$

Soln: it is parabola opening upword.

$$y = x^2-4x+4-1 = (x-2)^2-1$$

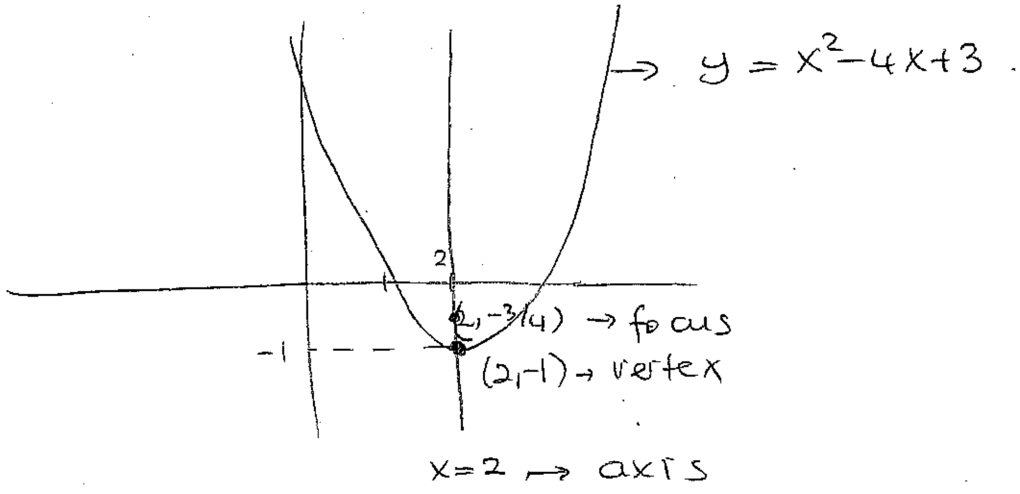
$y-(-1) = (x-2)^2 \longrightarrow$ parabola $y = x^2$
shifted 2 units right
1 unit down.

so, vertex $= (2,-1)$

axis : $x = 2$

For $y = x^2$ focus $= (0, \frac{1}{4})$ $y = \frac{x^2}{4p}$

For $y = x^2-4x+3$ focus $= (0+2, \frac{1}{4}-1) = (2, -3/4)$

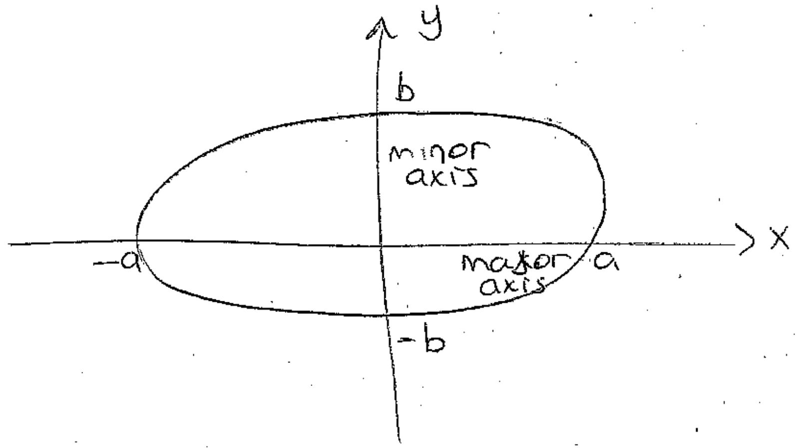


Ellipses and Hyperbolas:

Defn

If $a > 0$ and $b > 0$

$\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$ represents an ellipse



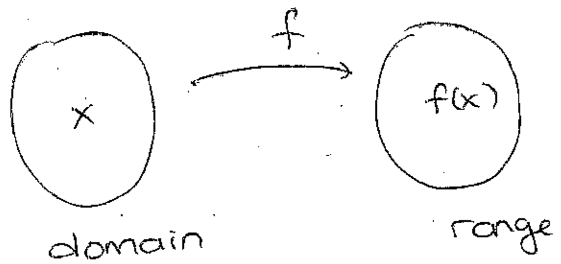
* It has centre at $(0, 0)$
* It passes through $(a, 0)$, $(-a, 0)$, $(0, b)$ and $(0, -b)$.
* The line segments from $(-a, 0)$ to $(a, 0)$ and from $(0, -b)$ to $(0, b)$ are called principal axes of ellipse.
* The longer principal axis is called major axis
  shorter " " minor axis

P.4 Functions and Their Graphs:

Defn: A function f on a set D into a set S is a rule that assigns a unique element $f(x)$ in S to each element x in D.

Defn: The domain of a function is the set of all possible input values.

Defn: The range of a function is the set of all output values



Ex: Let

$$F(t) = 2t+3.$$

find the output values of F that correspond to the input values, $0, 2, x+2, F(2)$.

Soln:

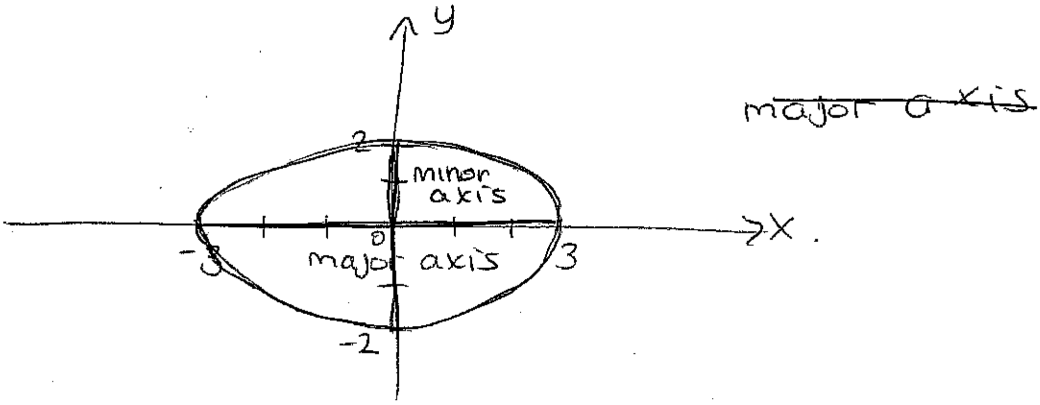
$$F(0) = 2(0)+3 = 3$$

$$F(2) = 2(2)+3 = 7$$

$$F(x+2) = 2(x+2)+3 = 2x+4+3 = 2x+7$$

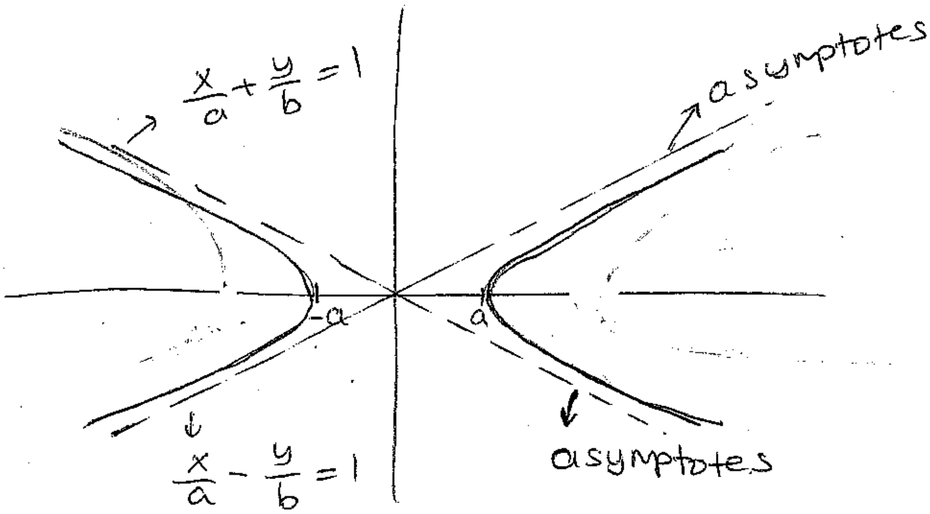
$$F(F(2)) = 2F(2)+3 = 2(7)+3 = 17$$

Ex $\frac{x^2}{9} + \frac{y^2}{4} = 1 \rightarrow$ ellipse.



Defn:

The eqn. $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ represents hyperbola



* It has centre at (0,0). & passes through (a,0) & (-a

The domain convention:

When a function f is defined without specifying its domain, we assume that the domain consists of all real numbers x for which the value $f(x)$ of the function is a real number.

Ex: $f(x) = \sqrt{x}$

Domain $f = [0, \infty)$, negative numbers do not have real square roots.

Ex: Find the domain of

$$f(x) = \frac{x}{x^2-4}$$

Soln: $x^2-4 = (x-2)(x+2)$.

domain $f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Ex: $S(t) = \sqrt{1-t^2}$

Soln: $1 - t^2 \geq 0$

		-1		1	
$1-t^2$	$-$	$\bullet$	$+$	$\bullet$	$-$

$(1-t)(1+t) \geq 0$

soln: $[-1, 1]$

domain of $S = [-1, 1]$

Ex: $f(x) = 1 + x$

domain of $f = \mathbb{R}$.

Ex: $g(x) = \dfrac{x}{\sqrt{2-x}}$

$\sqrt{x-2} \geq 0$

$x \geq 2$

Soln: $2 - x > 0$

$2 > x$

domain of $g = (-\infty, 2)$

Ex: Find the domain of

$$g(x) = \frac{1}{1 - \sqrt{x-2}}$$

$1 - \sqrt{x-2} = 0$

$\sqrt{x-2} = 1$

$x - 2 = 1$

$x = 3$

$\sqrt{x-2} \geq$

$x - 2 \geq$

$x \geq 2$

domain $= [2,3) \cup (3, +\infty)$

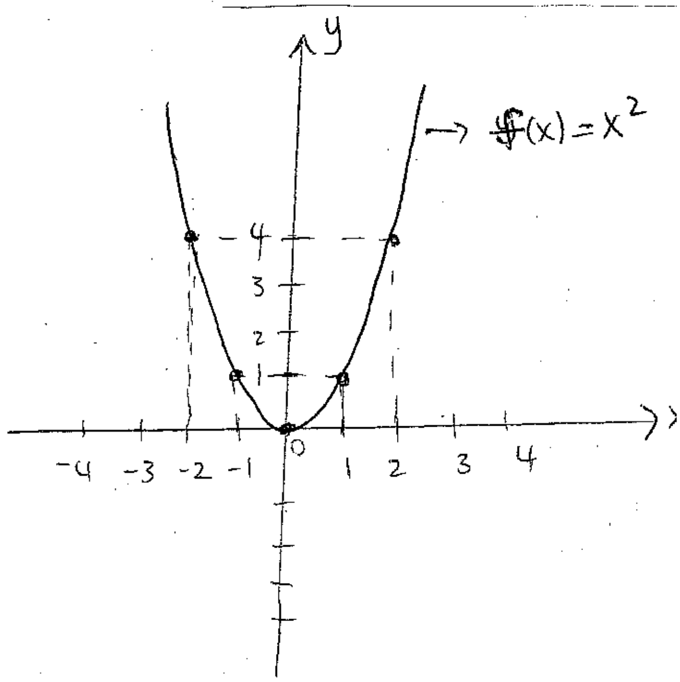
border

Graphs of Functions:

Ex: Graph the function $f(x) = x^2$

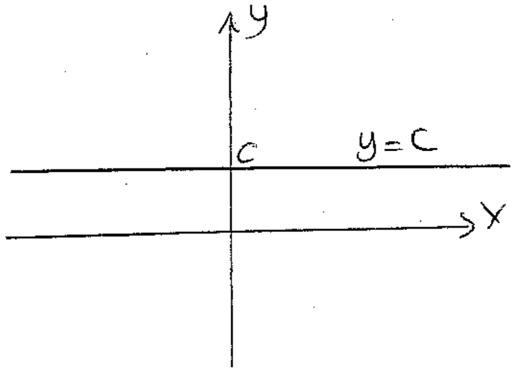
Soln:

x	$f(x) = x^2$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$

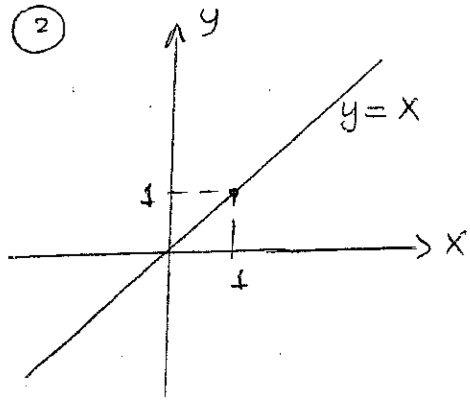


Graph of some functions:

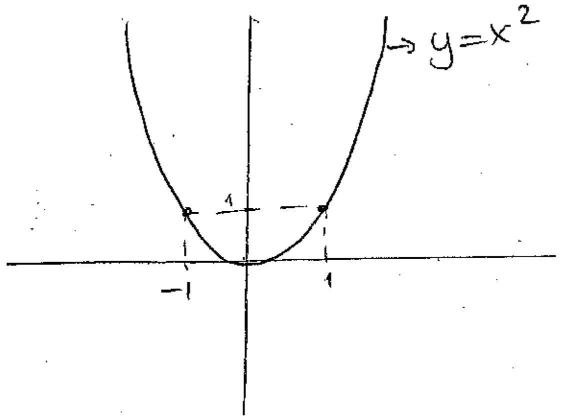
1.



2.



3.



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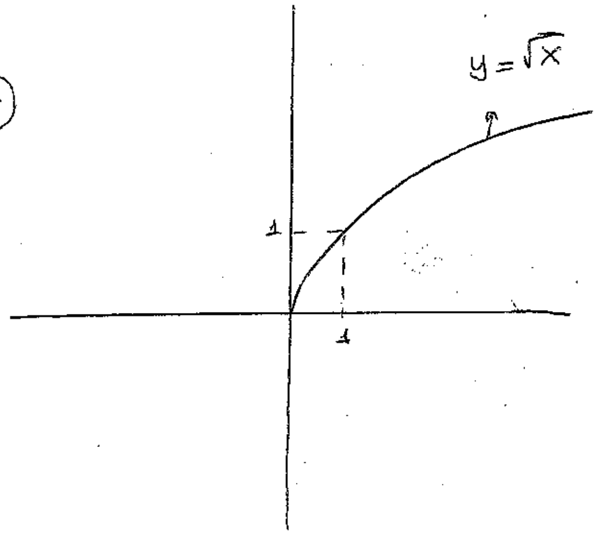

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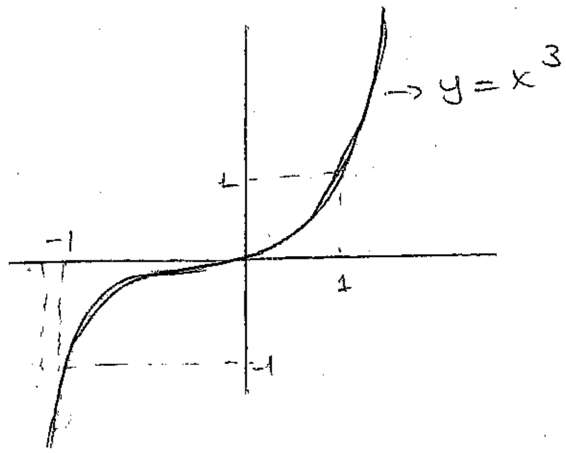

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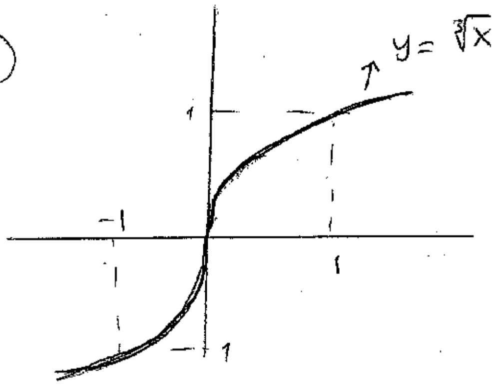

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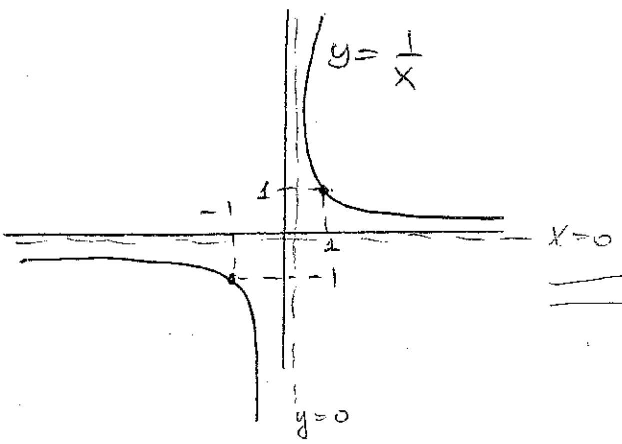
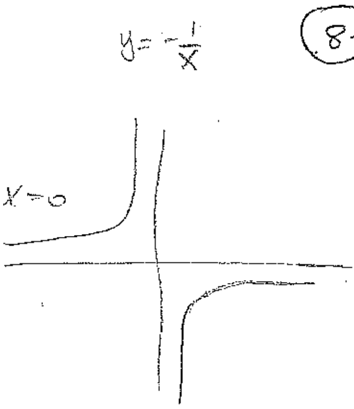
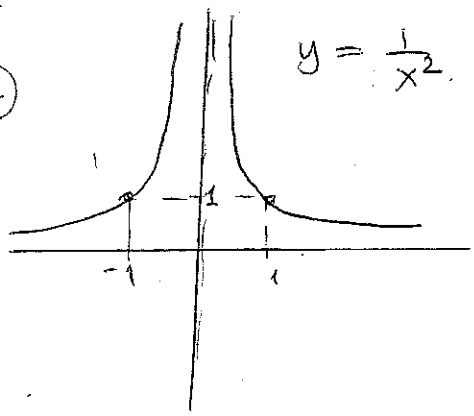
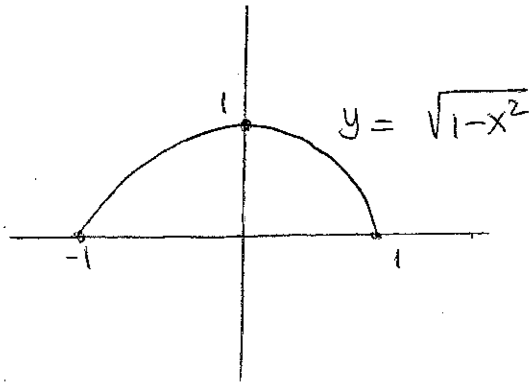




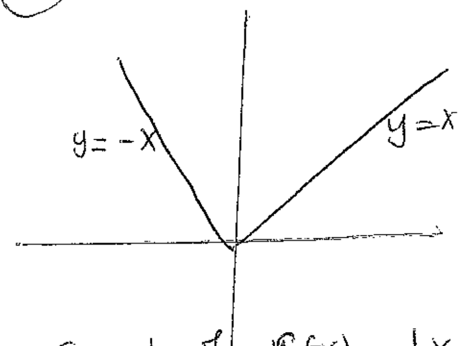
8.



(9)



(10.)

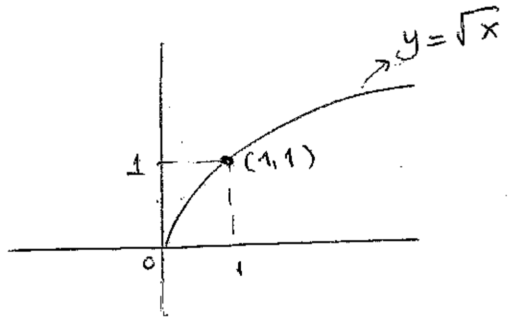


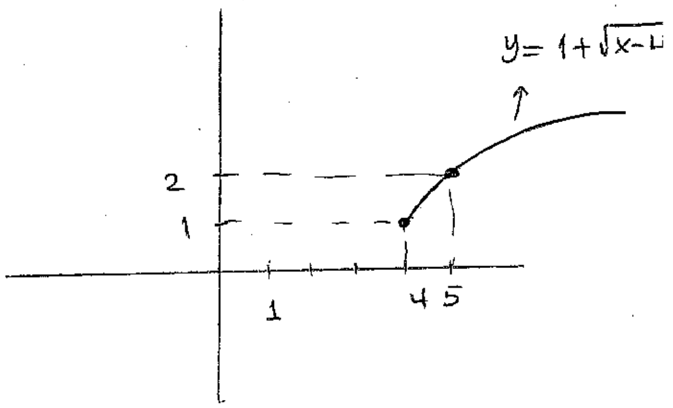
Graph of $f(x) = |x|$

Absolute value.

Ex 1 Sketch the graph of $y = 1 + \sqrt{x-4}$

Soln: This is graph of $y = \sqrt{x}$ shifted 4 units right and 1 unit up.





Ex: 2 Sketch the graph of the function

$$f(x) = \frac{2-x}{x-1}$$

Soln:

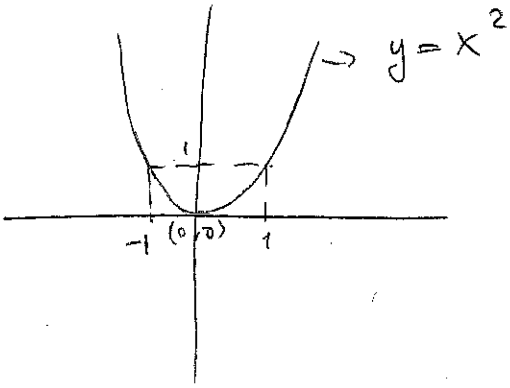
$$\frac{2-x}{x-1} = \frac{1+1-x}{x-1} = \frac{1-(-1+x)}{x-1} = \frac{1}{x-1} - 1$$

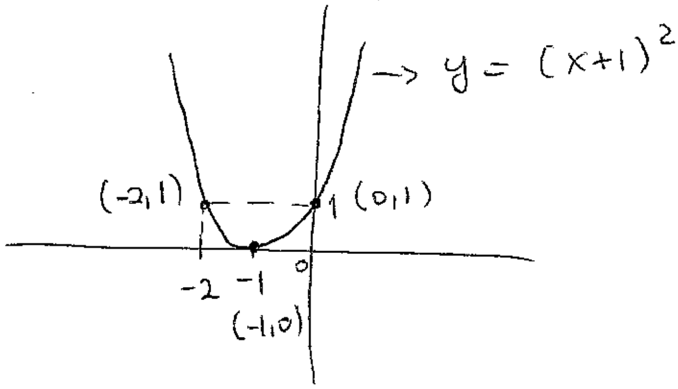
Graph of $\frac{1}{x}$ shifted 1 unit to the right and 1 unit dow

Q25

n^2 **Ex:3** Sketch the graph of $y = (x+1)^2$ $\qquad (x-(-1))^2$

Soln: It is graph of $y = x^2$, shifted 1 unit to **left**

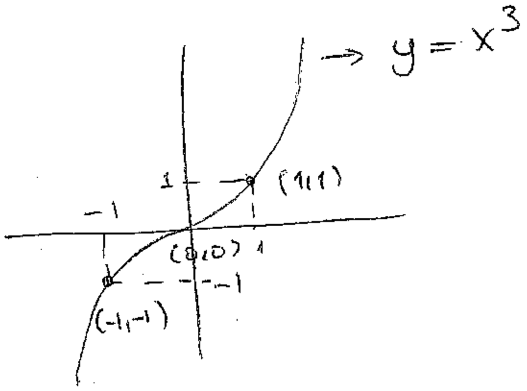


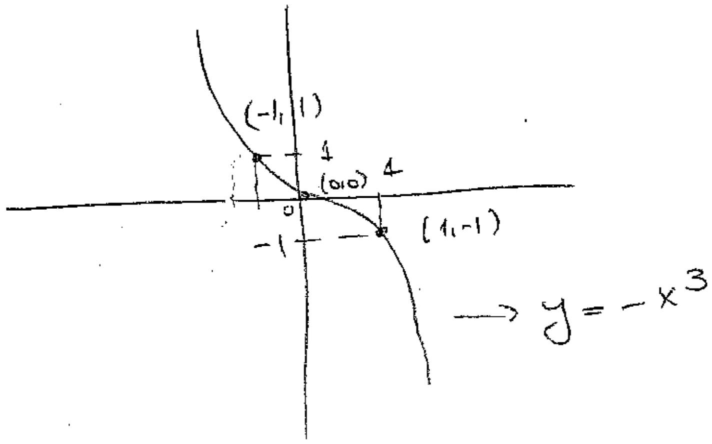


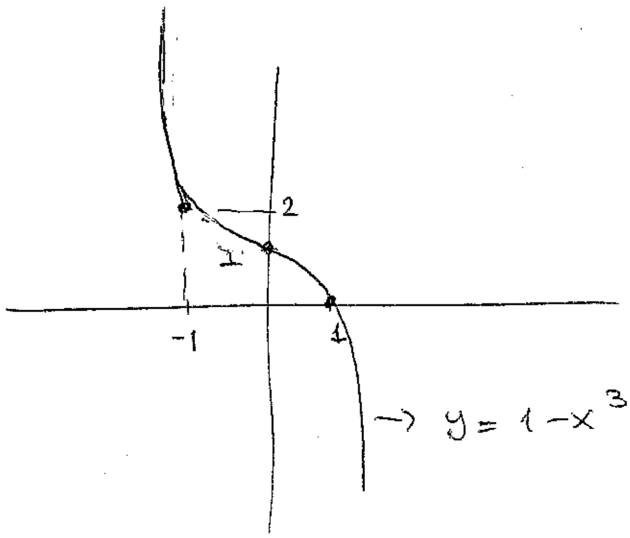
Q27

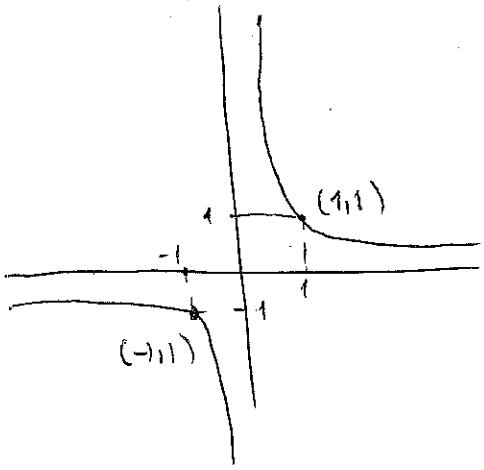
✓ **Ex4** sketch the graph of $y = 1 - x^3$

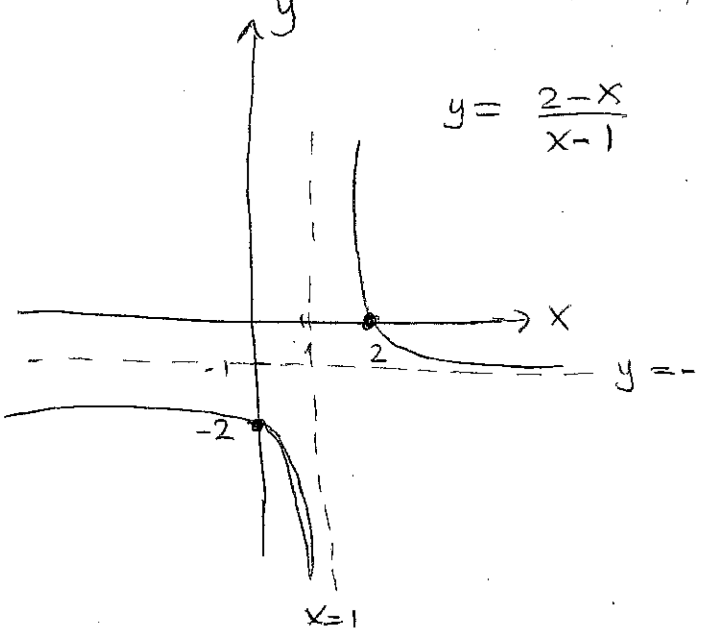
Soln: Same as graph of $y = -x^3$ shifted 1 unit up.







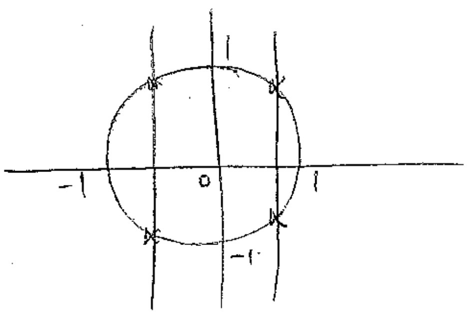




<u>Note:</u> A function f can have only one value f(x) for each x in its domain.

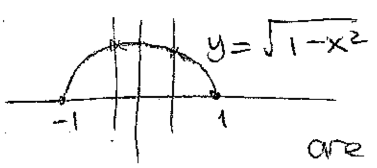
<u>Vertical Line Test:</u> Given graph is a <u>graph of a function</u> if every vertical line intersects the graph at most one point.

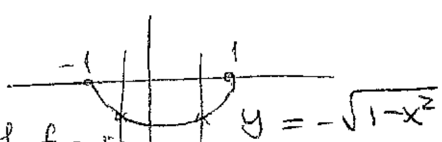
<u>Ex:</u> $x^2 + y^2 = 1$



not graph of a function

But. $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$





are graph of function

Even & Odd Functions ; Symmetry and Reflections

Defn: Let $-x \in$ domain of f whenever $x \in$ domain of f

then, f is an <u>even function</u> if

$$\boxed{f(-x) = f(x)} \quad \text{for } \forall x \in \text{domain of } f$$

f is <u>odd function</u> if

$$\boxed{f(-x) = -f(x)} \quad \text{for } \forall x \in \text{domain of } f$$

Ex: $x^0 = 1$, x^2, x^4, $\ldots$, x^{-2}, x^{-4}, $\ldots$ are even functions

x, x^3, $\cancel{x}\ldots$, x^{-1}, x^{-3}, $\ldots$ are odd functions.

because,

$$(-x)^2 = x^2 = (x)^2$$

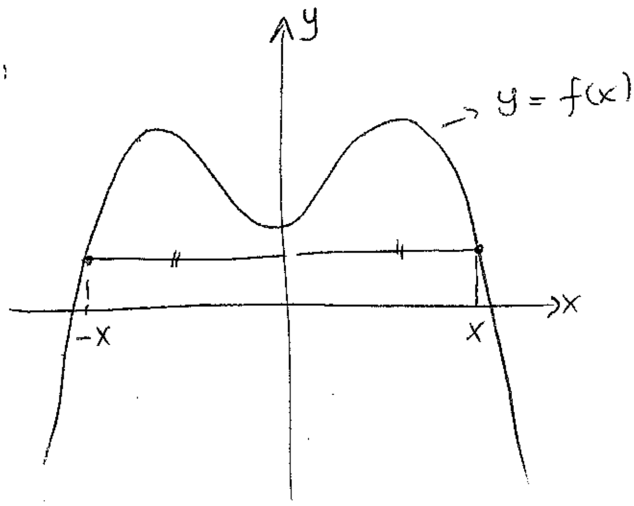
$$(-x)^3 = -x^3$$

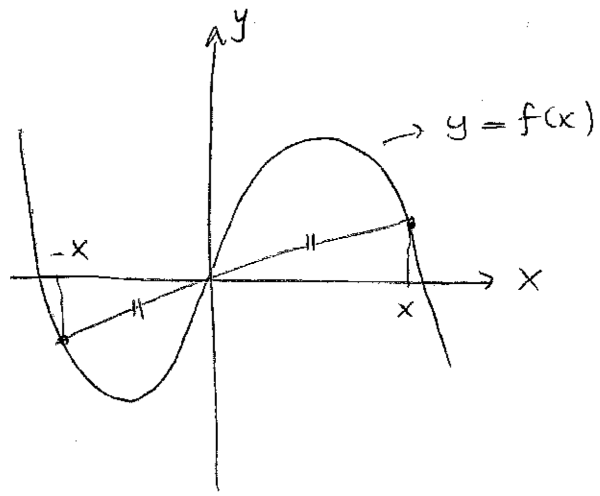
Ex: $y = |x| = \sqrt{x^2} \longrightarrow$ even

* The graph of an <u>even function</u> is <u>symmetric</u> about the <u>y-axis</u> Ex: $y = x^2$

* The graph of an <u>odd function</u> is <u>symmetric</u> about the <u>origin</u>. Ex: $y = x^3$

Ex:





Note:

* If $f(x)$ is odd or even $\Rightarrow$ $c f(x)$ is even or odd
$\Rightarrow$ ~~so is any~~ constant ~~multiple of f(x)~~.

Ex: $3 \cdot f(x)$, $-4f(x)$

If f, g are even or odd $\Rightarrow$ $f \mp g$ are even or odd

* Sum and differences of even functions are even

" " " odd " " odd

Ex: $f(x) = \underbrace{3x^4}_{\text{even}} - \underbrace{5x^2}_{\text{even}} - \underbrace{1}_{\text{even}}$ $1 = x^0$ $\longrightarrow$ even

Ex $f(x) = \underbrace{4x^3}_{\text{odd}} - \underbrace{\left(\frac{2}{x}\right)}_{\text{odd}} = -2x^{-1}$ $\longrightarrow$ odd

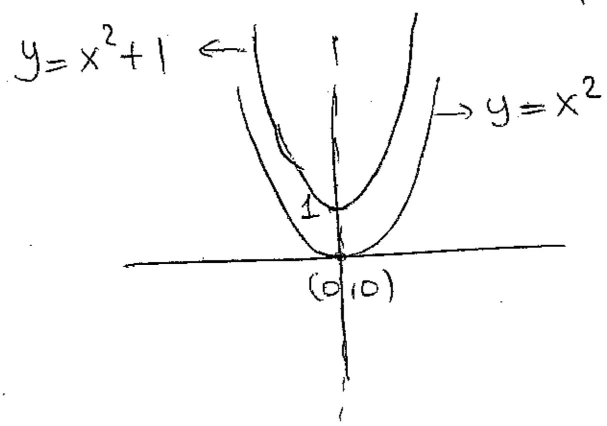
Ex: $f(x) = \underbrace{x^2}_{\text{even}} - \underbrace{2x}_{\text{odd}}$ $\rightarrow$ neither even nor odd.

Example: For the following functions, what (if any) symmetry does the graph of f posses? In particular is f either even or odd?

ex 11 p. 34 a.) $f(x) = x^2 + 1$

$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$, So f is <u>even</u> functi

$f(x) = x^2 + 1 \rightarrow$ has same graph as $y = x^2$, shifted 1 unit up.



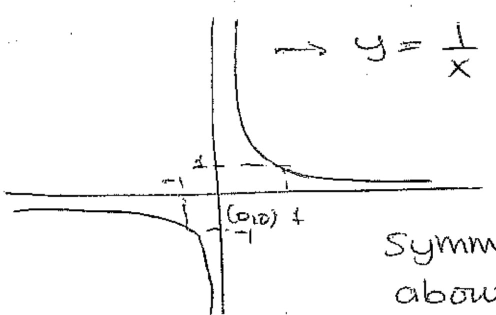
symmetric about y-axis.

b.) $f(x) = \dfrac{1}{x-2}$

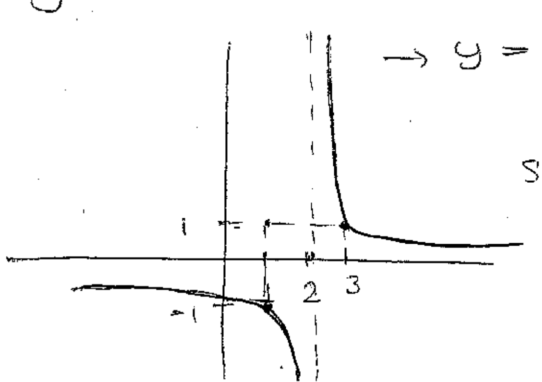
$f(-x) = \dfrac{1}{-x-2}$ neither even nor odd.

$f(x) = \dfrac{1}{x-2}$ has same graph as $y = \dfrac{1}{x}$

shifted 2 units to right.



Symmetric about (0,0)

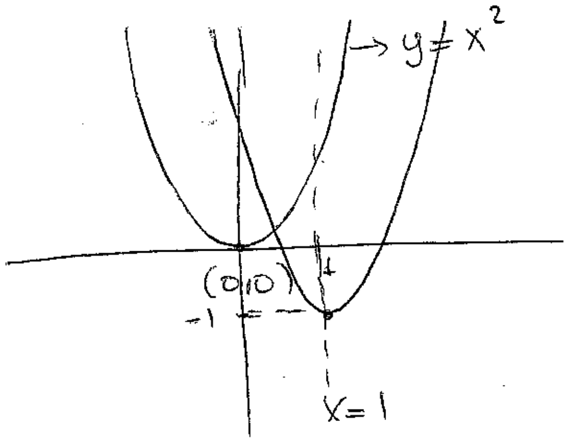


symmetric about (2,0)

Note: There are also other kinds of symmetry.

Ex: Let $g(x) = x^2 - 2x$

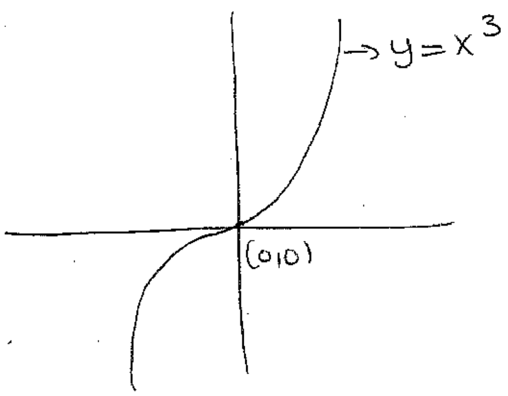
$g(x) = (x-1)^2 - 1 \longrightarrow$ same as graph of $y = x^2$ shifted 1 unit to the right and 1 unit down.



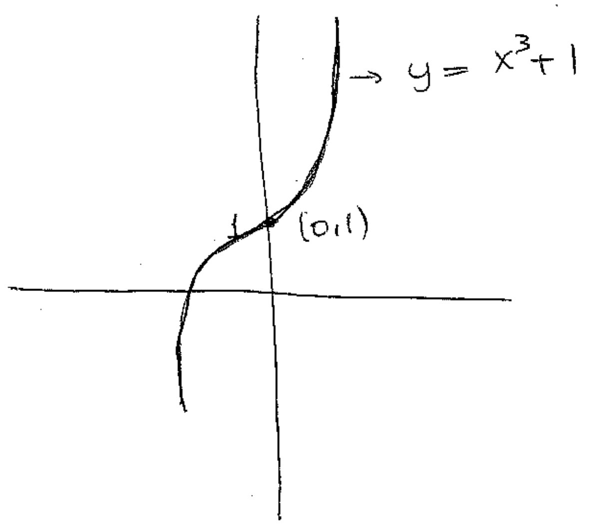
The graph of $g(x)$ is symmetr about the vertical line $x = 1$.

Ex: Let $h(x) = x^3 + 1$

$h(x)$ has same graph as $y = x^3$ shifted 1 unit up.



symmetric about (0,0)

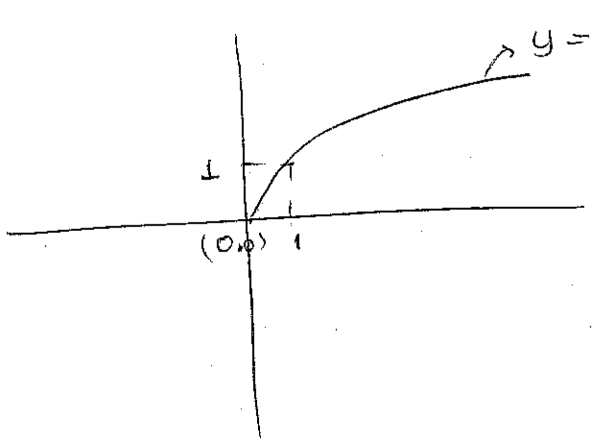


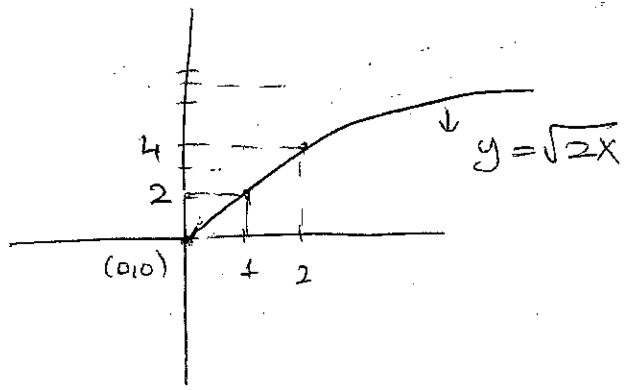
symmetric about (0,1).

c) $f(x) = \sqrt{2x}$

neither ~~even~~ nor odd.

has same graph as $\sqrt{x}$





No symmetry.

Reflections in Straight Lines

Defn: Given a line L and a point P not on L, we call a point Q the reflection of P in L if L is the right bisector of the line segment PQ.

Certain reflections of graphs are easily described in terms of the eqns of the graphs:

1.) Subst $-x$ in place of x in an eqn in x and y
$\Rightarrow$ reflection of graph of eqn in y-axis

2.) Subst. $-y$ in place of y in an eqn in x and y
$\Rightarrow$ reflection in x-axis

3.) Subst. $\underline{a-x}$ in place of x in an eqn in x and y

$\Rightarrow$ reflection of graph in the line $\underline{x = a/2}$

4.) Subst. $\underline{b-y}$ in place of y in an eqn in x and y

$\Rightarrow$ reflection of graph in the line $\underline{y = b/2}$

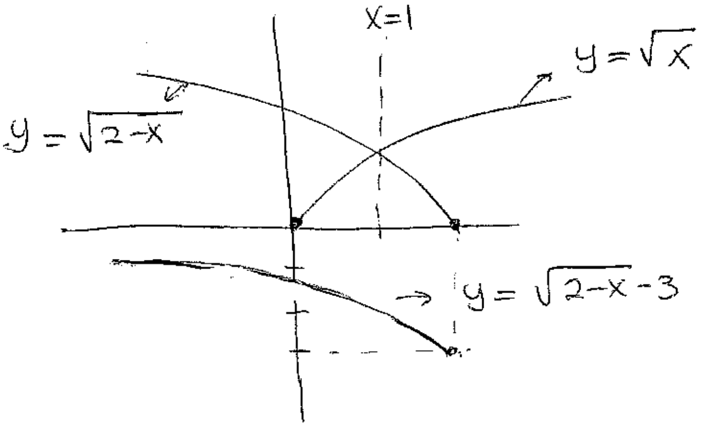
5.) Interchanging $\underline{x}$ and $\underline{y}$ in an eqn in x and y

$\Leftrightarrow$ reflection of graph in the line $\underline{y = x}$.

Ex Describe and sketch the graph of $y = \sqrt{2-x} - 3$

Soln: graph of $y = \sqrt{2-x}$ is reflection of graph of $y = \sqrt{x}$

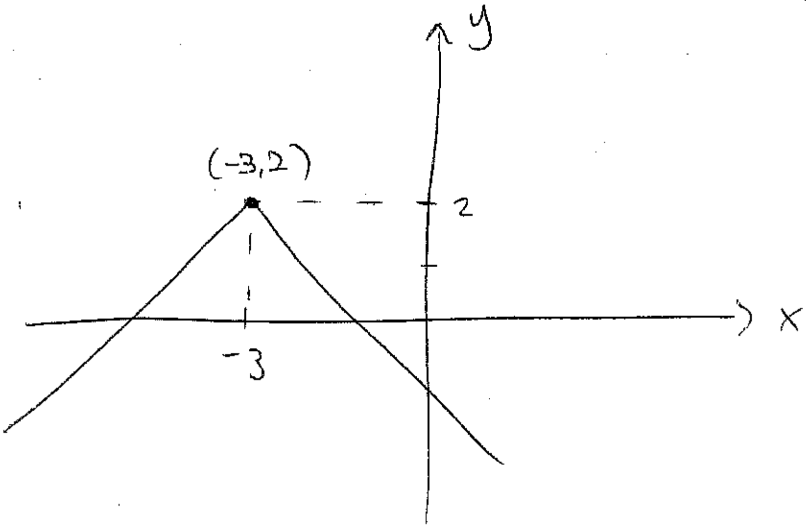
in the line $x = \frac{2}{2} = 1$



graph of $y = \sqrt{2-x} - 3$

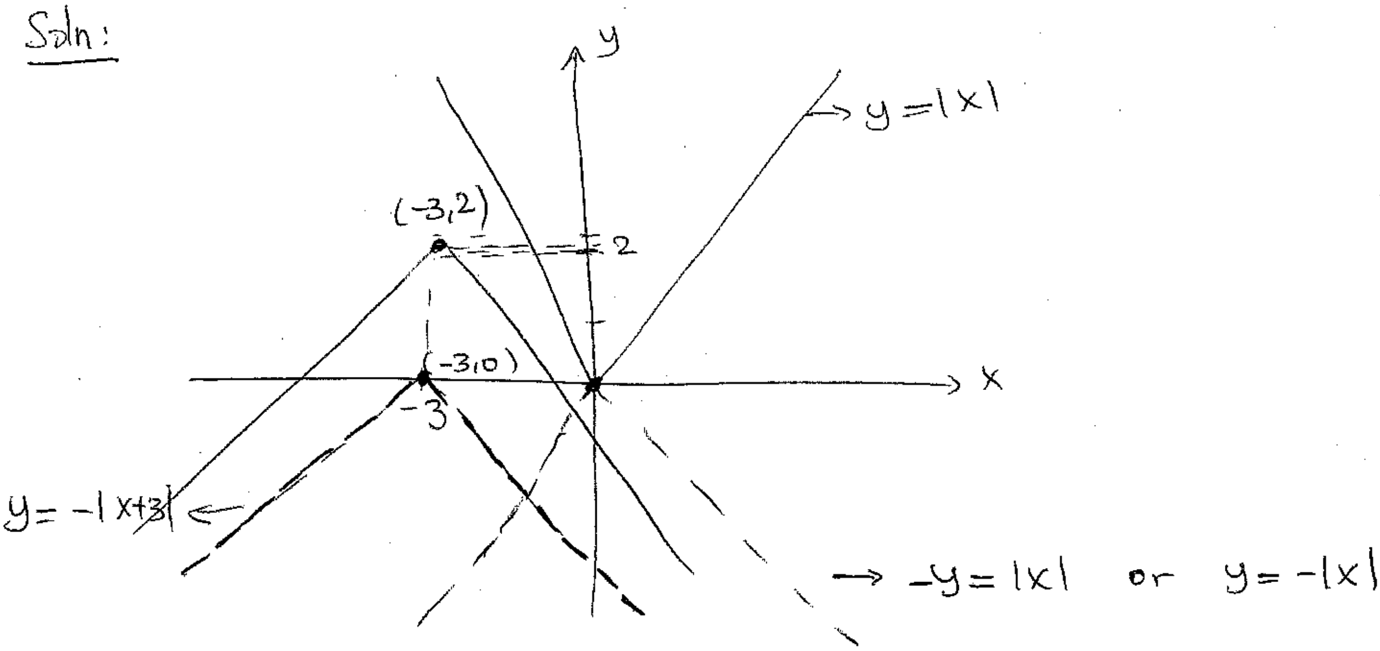
lowering reflection 3 units down.

Ex:



Express the eqn of the graph in terms of ~~of~~ $|x|$

Soln:



1.) Reflect $y=|x|$ in the x-axis

2.) shift reflected graph 3 units left

3.) then shifting 2 units up.

P.5 Combining Functions To Make New Functions

Sums, Differences, Products, Quotients and Multiples

Defn: If, f and g are functions, then for $\forall\ x \in \operatorname{dom} f$ and $x \in \operatorname{dom} g$

a.) $(f+g)(x) = f(x) + g(x)$

b.) $(f-g)(x) = f(x) - g(x)$

c.) $(fg)(x) = f(x)\,g(x)$

d.) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

e.) $(cf)(x) = c f(x)$

Ex: Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$

Find formulas for.

a.) $3f$ d.) fg

b.) $f+g$ e) f/g

c.) $f-g$ f) g/f

and specify the domains of each of these functions.

Soln: a.) $3f = (3f)(x) = 3\sqrt{x}$

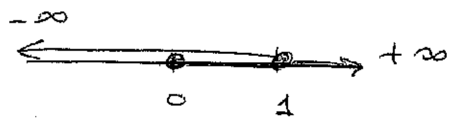
domain $= [0, \infty)$

b.) $f+g = f(x)+g(x) = \sqrt{x} + \sqrt{1-x}$

domain of $f+g = [0,1]$

$x \geq 0$ and $1-x \geq 0$

$1 \geq x.$ $\Rightarrow$



c.) $f-g = f(x)-g(x) = \sqrt{x} - \sqrt{1-x}$

domain of $f-g = [0,1]$

d.) $fg = f(x)\,g(x) = \sqrt{x}\,\sqrt{1-x} = \sqrt{x-x^2}$

$x - x^2 \geq 0$

$x(1-x) \geq 0$

x		0		1	
$x-x^2$	$-$		$+$		$-$

domain of $fg = [0,1]$

e.) $f/g = \dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x}}{\sqrt{1-x}} = \sqrt{\dfrac{x}{1-x}}$

$\dfrac{x}{1-x} \geq 0.$

$x = 0$

$1-x=0 \Rightarrow x=1$

x		0		1	
x	$-$		$+$		$+$
$1-x$	$+$		$+$		$-$
$\dfrac{x}{1-x}$	$-$		$+$		$-$

domain of $\dfrac{f}{g} = [0,1)$

f.) $\dfrac{g}{f} = \dfrac{g(x)}{f(x)} = \dfrac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\dfrac{1-x}{x}}$

$\dfrac{1-x}{x} \geq 0$

x		0		1	
$1-x$	$+$		$+$	0	$-$
x	$-$	0	$+$		$+$
$\frac{1-x}{x}$	$-$		$+$		$-$

domain of $\dfrac{g}{f} = (0, 1]$

Composite Functions :

Defn: If f and g are two functions, the composite function $f \circ g$ is defined by

$$\boxed{f \circ g(x) = f(g(x))}$$

Ex: Let $f(x) = \sqrt{x}$ and $g(x) = x+1$, calculate

a.) $f \circ g(x)$

b.) $g \circ f(x)$

c.) $f \circ f(x)$

d.) $g \circ g(x)$

and specify the domain of each.

Soln: a.) $f \circ g(x) = f(x+1) = \sqrt{x+1}$

domain of $f \circ g(x)$: $[-1, \infty)$

$x+1 \geq 0$

$x \geq -1$

b.) $g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x}+1$

$x \geq 0$. domain of $g \circ f(x) = [0, \infty)$

c.) $f \circ f(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4} = \sqrt[4]{x}$

$x \geq 0$ domain of $f \circ f(x) = [0, \infty)$.

d.) $g \circ g(x) = g(g(x)) = g(x+1) = x+1+1 = x+2$

domain of $g \circ g(x) = \mathbb{R}$.

Ex: Let $G(x) = \frac{1-x}{1+x}$. calculate $G \circ G(x)$ and specify its domain.

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Soln: $G \circ G(x) = G(G(x)) = \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{\frac{\cancel{1}+x-\cancel{1}+x}{1+x}}{\frac{\cancel{1}+\cancel{x}+1-\cancel{x}}{1+x}} = \frac{\cancel{2}x}{\cancel{1+x}} \cdot \frac{\cancel{1+x}}{2}$

$= x$

For domain $G \circ G$,

1.) x must be in domain of G $\Rightarrow$ $x \neq -1$

2.) $G(x)$ must belong to the domain of G.

$G(x) \in D_G \Rightarrow G(x) \neq -1$

domain of $G = \mathbb{R} - 1$.

$\frac{1-x}{1+x} = -1$

$G(x) = -1$ has no soln.

has no soln.

domain of $G \circ G = (-\infty, -1) \cup (-1, \infty) = \mathbb{R} - \{-1\}$.

Ex Let $f(x) = \frac{2}{x}$ and $g(x) = \frac{x}{1-x}$ Find dom of $f \circ g$ and $f \circ g$.

$f \circ g(x) = \frac{2}{\frac{x}{1-x}} = \frac{2-2x}{x}$. if $x \in D_g \Rightarrow x \neq 1$.

if $g(x) \in D_f \Rightarrow \frac{x}{1-x} \neq 0 \Rightarrow x \neq 0$.

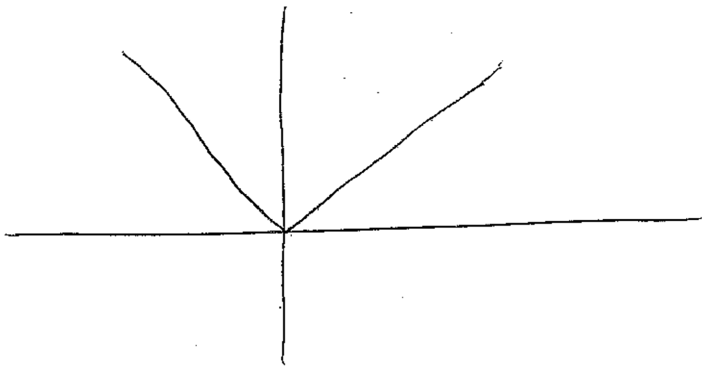
$D_{f \circ g} = \mathbb{R} - \{0, 1\}$

Piecewise Defined Functions:

These are the functions which are defined by using different formulas on different parts of its domain.

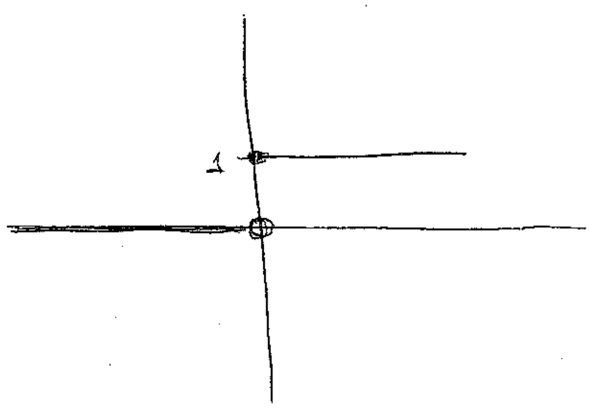
Ex

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



Ex: Heavisicle function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

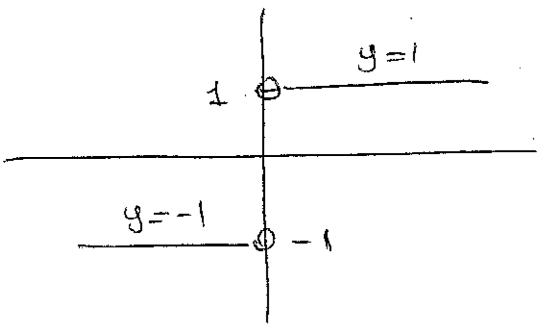


it is used to model voltage applied to an electric circui by one volt battery if a switch in the circuit is closed at $t =$

Ex: The Signum function

$$\operatorname{Sgn}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

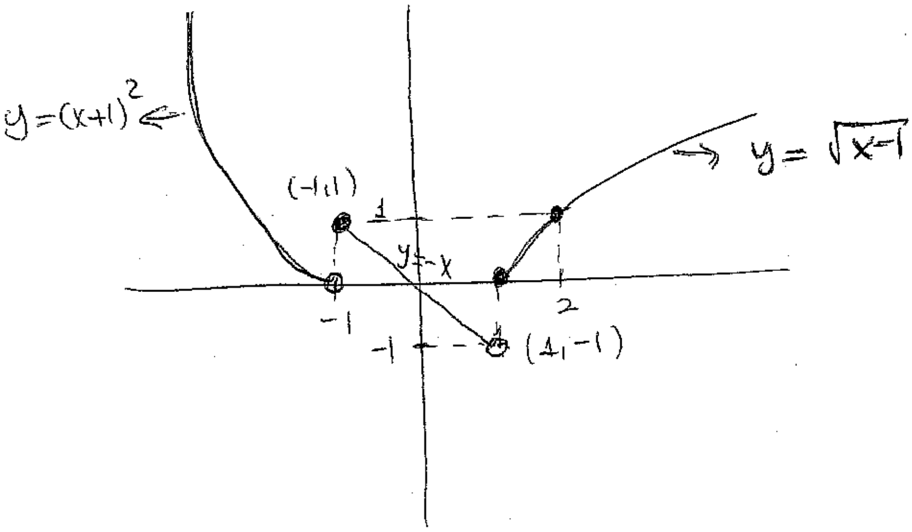
Sgn(x) is an odd function.



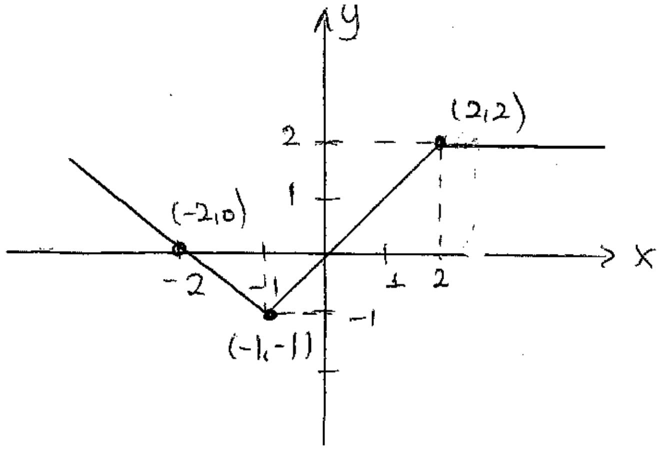
Ex:

$$f(x) = \begin{cases} (x+1)^2 & \text{if } x < -1 \\ -x & \text{if } -1 \le x < 1 \\ \sqrt{x-1} & \text{if } x \ge 1 \end{cases}$$

Soln:



Ex: Find a formula for function



Soln:

$$m = \frac{-1-0}{-1+2} = \frac{-1}{1} = -1$$

$y - 0 = -1\,(x+2) \Rightarrow y = -(x+2)$, when $x < -1$

$y = x$ when $-1 \le x \le 2$

$y = 2$ for $x > 2$.

So,

$$f(x) = \begin{cases} -(x+2) & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 2 \\ 2 & \text{if } x > 2 \end{cases}$$

Ex:

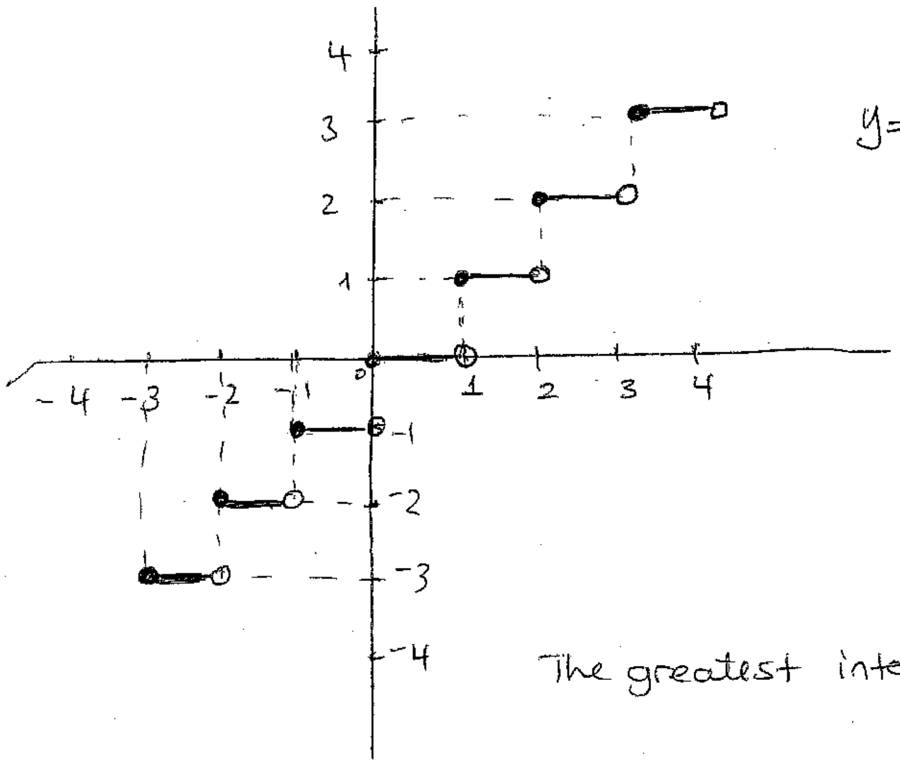
Defn: The greatest Integer Function:

It is the function whose value at any number x is the greatest integer less than or equal to x

It is denoted $\lfloor x \rfloor$ or $[x]$ or $[[x]]$

$\lfloor 2.4 \rfloor = 2$ $\quad \lfloor 1.9 \rfloor = 1$ $\quad \lfloor 0 \rfloor = 0$ $\quad \lfloor -1.2 \rfloor = -2$

$\lfloor 2 \rfloor = 2$ $\quad \lfloor 0.2 \rfloor = 0$ $\quad \lfloor -0.3 \rfloor = -1$ $\quad \lfloor -2 \rfloor = -2$



The greatest integer function $\lfloor x \rfloor$

Ex: Defn: The Least Integer Function

It is the function whose value at any number x is the smallest integer greater than or equal to x

It is denoted $\lceil x \rceil$