c.) $u = \frac{3}{5} x^{5/3} - \frac{5}{3} x^{-3/5}$

$$\frac{du}{dx} = \frac{\cancel{3}}{\cancel{5}} \cdot \frac{\cancel{5}}{3} x^{2/3} - \frac{\cancel{5}}{\cancel{3}} \cdot \frac{-\cancel{3}}{\cancel{5}} x^{-8/5}$$

$$\frac{du}{dx} = x^{2/3} + x^{-8/5}$$

<u>Ex:</u> Find an eqn of the tangent line to $y = \frac{1-4x^2}{x^3}$ at the point on the curve where $x = 1$

$$\left.\frac{dy}{dx}^*\right|_{x=1} = \frac{d}{dx}\left.\left(\frac{1-4x^2}{x^3}\right)\right|_{x=1} = \frac{d}{dx}\left.\left(\frac{1}{x^3} - \frac{4x^2}{x^3}\right)\right|_{x=1}$$

$$= \frac{d}{dx}\left.\left(\frac{1}{x^3} - \frac{4}{x}\right)\right|_{x=1} = \left.\left(\frac{-3}{x^4} + \frac{4}{x^2}\right)\right|_{x=1} = 1$$

$m = 1$ // slope of tangent line

if $x = 1 \Rightarrow y = \frac{1-4}{1} = \frac{-3}{1} = -3$ $\quad (1, -3)$.

Eqn of tangent line to curve at $(1, -3)$ is:

$$y + 3 = 1\,(x - 1) \Rightarrow \boxed{y = x - 4}$$

<u>Thm3</u> Product Rule

$$\boxed{(fg)'(x) = f'(x)\,g(x) + g'(x)\,f(x)}$$

Ex: Find the derivative of $(2x^2+2)(x+4)$ using and without using the Product Rule.

Soln:

$f(x) = (2x^2+2) \qquad g(x) = (x+4)$

$(fg)'(x) = 4x(x+4) + (2x^2+2)\cdot 1$

$= 4x^2 + 16x + 2x^2 + 2 \quad = 6x^2 + 16x + 2$ //

without using,

$\cancel{k}(x) = (2x^2+2)(x+4) = 2x^3 + 8x^2 + 2x + 8$

$k'(x) = 6x^2 + 16x + 2$ //.

Ex: Find $y'(x)$ if $y(x) = (x^2+4)(\sqrt{x}+1)$

$y'(x) = 2x(\sqrt{x}+1) + (x^2+4)\left(\frac{1}{2\sqrt{x}}\right)$

$= 2x^{3/2} + 2x + \frac{2}{\sqrt{x}} + \frac{1}{2}x^{3/2}$

$= \frac{5}{2}x^{3/2} + 2x + \frac{2}{\sqrt{x}}$

Ex: Use mathematical induction to verify

$$\frac{d}{dx} x^n = n x^{n-1} \quad \text{for all positive integers } n$$

Soln:

For $n=1$ $\quad \frac{d}{dx} x^1 = 1 x^0 = 1$ true.

Show that if formula is true for $n = k \geq 1$, then it is also true for $n = k+1$

assume $\quad \frac{d}{dx} x^k = k x^{k-1}$

$$\frac{d}{dx} x^{k+1} = \frac{d}{dx} (x^k \cdot x) = \left(\frac{d}{dx} x^k\right)(x) + \frac{d}{dx}(x) \cdot x^k$$

use product rule.

$$= k x^{k-1} \cdot x^1 + 1 \cdot x^k$$

$$= k x^k + x^k$$

$$\frac{d}{dx} x^{k+1} = (k+1) x^k$$ formula is true for $n = k+1$

So formula is true for all integers $n \geq 1$ by induction.

Remark: Product Rule can be extended to products of any number of factors

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'$$

Bader

Thm: Reciprocal Rule

If f is differentiable at x and $f(x) \neq 0 \Rightarrow \frac{1}{f}$ is differentiable at x

$$\boxed{\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}}$$

Ex: a.) $f(x) = \dfrac{1}{x+\sqrt{x}}$

$$f'(x) = \frac{-\left(1+\frac{1}{2\sqrt{x}}\right)}{(x+\sqrt{x})^2} = -\frac{(2\sqrt{x}+1)}{2\sqrt{x}\,(x+\sqrt{x})^2}$$

b.) $f(x) = \dfrac{1}{x^2+5x}$

$$f'(x) = \frac{-(2x+5)}{(x^2+5x)^2}$$

General Power Rule:

$$\boxed{\frac{d}{dx}x^{-n} = -n\,x^{-n-1}}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n\,x^{n-1}}{(x^n)^2} = -n\,x^{-n-1}$$

Ex: $y = \dfrac{x-1}{x^{2/3}}$

$$\frac{d}{dx}\left(\frac{x-1}{x^{2/3}}\right) = \frac{d}{dx}\left(\frac{x}{x^{2/3}}\right) - \frac{d}{dx}\left(\frac{1}{x^{2/3}}\right)$$

$$= \frac{d}{dx}\left(x^{1/3}\right) - \frac{d}{dx}\left(x^{-2/3}\right)$$

$$= \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3}$$

$$= \frac{1}{3x^{2/3}} + \frac{2}{3x^{5/3}}$$

Thm: Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$

then, $\frac{f}{g}$ is differentiable at x.

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}}$$

Ex: a) $z = \dfrac{t^2+2t}{t^2-1}$

$$\frac{dz}{dt} = \frac{(2t+2)(t^2-1) - (2t)(t^2+2t)}{(t^2-1)^2}$$

$$= \frac{\cancel{2t^3} + 2t^2 - 2t - 2 - \cancel{2t^3} - 4t^2}{(t^2-1)^2}$$

$$\frac{dz}{dt} = \frac{-2t^2 - 2t - 2}{(t^2-1)^2}$$

b.) $s = \dfrac{1+\sqrt{t}}{1-\sqrt{t}}$

$$\frac{ds}{dt} = \frac{\left(\frac{1}{2\sqrt{t}}\right)(1-\sqrt{t}) - (1+\sqrt{t})\left(-\frac{1}{2\sqrt{t}}\right)}{(1-\sqrt{t})^2}$$

$$= \frac{\frac{1}{2\sqrt{t}} - \cancel{\frac{1}{2}} + \frac{1}{2\sqrt{t}} + \cancel{\frac{1}{2}}}{(1-\sqrt{t})^2} = \frac{\frac{\cancel{2}}{\cancel{2}\sqrt{t}}}{(1-\sqrt{t})^2} = \frac{1}{\sqrt{t}(1-\sqrt{t})^2}$$

<u>Ex:</u> Find eqns of any lines that pass through the point $(-1,0)$ and are tangent to the curve

$$y = \frac{x-1}{x+1}$$

<u>Soln:</u> $(-1,0)$ not on the curve $\Rightarrow$ it is not point of tangency.

Let line be tangent to curve at $x=a$.

$\Rightarrow$ point of tangency $\left(a, \frac{a-1}{a+1}\right)$

slope of tangent line $= y'(x) = \dfrac{1(x+1) - 1(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$.

$$y'(a) = \frac{2}{(a+1)^2}$$

Also, line passes through $(-1,0)$ so

$$\text{Slope} = \frac{\frac{a-1}{a+1} - 0}{a+1} = \frac{a-1}{(a+1)^2}$$

$\Rightarrow \quad \dfrac{a-1}{(a+1)^2} = \dfrac{2}{(a+1)^2} \quad \Rightarrow \quad \cancel{(a+1)^2}(a-1) = 2\cancel{(a+1)^2}$

$a = 2+1$

$a = 3.$

$\text{slope} = \dfrac{2}{(3+1)^2} = \dfrac{1}{8}$

There is only one line through $(-1,0)$ tangent to curve.

$y - 0 = \frac{1}{8}(x+1) \quad \Rightarrow \quad \boxed{y = \frac{1}{8}x + \frac{1}{8}}$

Sec 2-4: The Chain Rule

→ how to differentiate composites of functions whose derivatives are already known.

Thm: Chain Rule

If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x,

then $f \circ g(x) = f(g(x))$ is differentiable at x

$$\boxed{(f\circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

OR if $y = f(u) \quad u = g(x) \quad \Rightarrow \quad y = f(g(x)) = f\circ g(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.}$$

Ex: ~~let~~ ~~f~~

Find the derivative of $\frac{1}{x^2-4}$

Soln: $f(u) = \frac{1}{u}$ $\quad g(x) = x^2-4$

$$f\circ g(x) = f(g(x)) = \frac{1}{x^2-4}$$

$$(f\circ g)'(x) = f'(g(x))\cdot g'(x) = \left(\frac{-1}{(x^2-4)^2}\right)\cdot 2x = \frac{-2x}{(x^2-4)^2}$$ //

$f'(u) = -\frac{1}{u^2}$

$g'(x) = 2x$

Ex: Find the derivative of $y = \sqrt{2x^2-4}$

Soln:

$$y = f(g(x)) = \sqrt{2x^2-4}$$

$f(u) = \sqrt{u}$

$g(x) = 2x^2-4$

$$y' = (f\circ g)'(x) = f'(g(x))\cdot g'(x)$$

$\Rightarrow f'(u) = \frac{1}{2\sqrt{u}}$

$g'(x) = 4x$

$$= \frac{1}{2\sqrt{2x^2-4}}\cdot 4x$$

$$y' = \frac{2x}{\sqrt{2x^2-4}}$$

Note: Usually when applying chain Rule we do not introduce symbo[ls] to represent the functions being composed.

Ex: Find the derivative of: $g'(1)f(1)(2x^2-3x+1)^2(3x+4)^4$

(a.) $f(x) = (4-x^2)^{10}$

(b.) $y = |1-x^2|$

c.) $z = \left(u + \frac{1}{u-1}\right)^{-5/3}$

e.) $y = \tan 3u$, $u = x^2$, $\frac{dy}{dx} = ?$

$y = \left(u + \frac{1}{u-1}\right)^{-5/3}$, $u = x$, $\frac{dy}{dx} = ?$

Soln:

(a.) $f'(x) = 10(4-x^2)^9 \cdot (-2x) = -20x(4-x^2)^9$

(b.) $y' = \frac{1-x^2}{|1-x^2|} \cdot (-2x) = (-2x)\,\mathrm{Sgn}(1-x^2)$

$f(x) = |x|$

$f'(x) = \frac{x}{|x|} = \mathrm{Sgn}\, x$

(c.) $z' = \frac{-5}{3}\left(u + \frac{1}{u-1}\right)^{-8/3} \cdot \frac{d}{du}\left(u + \frac{1}{u-1}\right)$

$= \frac{-5}{3}\left(u + \frac{1}{u-1}\right)^{-8/3}\left(1 - \frac{1}{(u-1)^2} \cdot \frac{d}{du}(u-1)\right)$

$= \frac{-5}{3}\left(u + \frac{1}{u-1}\right)^{-8/3}\left(1 - \frac{1}{(u-1)^2}\right)$

Ex: Suppose f is differentiable function on $\mathbb{R}$. In terms of the derivative f' of f, express the derivatives of :

(a.) $f(x^2)$

$$\frac{d}{dx} f(x^2) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2) \cdot 2x$$

(b.) $f(\pi f(x))$

$$\frac{d}{dx} f(\pi f(x)) = (f'(\pi f(x))) \cdot (\pi f'(x))$$

(c.) $[f(3-2f(x))]^4$.

$$\frac{d}{dx}[f(3-2f(x))]^4 = 4[f(3-2f(x))]^3 \cdot f'(3-2f(x)) (-2f'(x))$$

$$= -8 f'(x) f'(3-2f(x)) [f(3-2f(x))]^3$$

Building The Chain Rule into Differentiation Formu

If u is diff function of x and $y = u^n$

By Chain Rule $\Longrightarrow$ $\frac{d}{dx} u^n = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = n u^{n-1} \cdot \frac{du}{dx}$.

$$\boxed{\frac{d}{dx} u^n = n u^{n-1} \cdot \frac{du}{dx}}$$

Ex: $y = (2x+3)^6$ $\qquad \frac{dy}{dx} = 6(2x+3)^5 \cdot 2 = 12(2x+3)^5$.

$$\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{-1}{u^2} \cdot \frac{du}{dx}. \qquad \text{(Reciprocal Rule)}$$

$$\frac{d}{dx}\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}. \qquad \text{(Square Root Rule)}$$

$$\frac{d}{dx}|u| = \operatorname{sgn} u \cdot \frac{du}{dx} = \frac{u}{|u|} \cdot \frac{du}{dx}. \qquad \text{(Absolute Value Rule)}$$

Sec 2.5 Derivatives of Trigonometric Functions:

Sine, Cosine funct → used in mathematical modelling.

They arise when quantities fluctuate in a periodic way. } Ex: vibrations, Elastic motions, waves

Physical & mechanical laws formulated as diff eqns having These funct as soln.

Thm: $\lim_{\theta \to 0} \sin\theta = \sin 0 = 0$ $\qquad \lim_{\theta \to 0} \cos\theta = \cos 0 = 1$

Thm: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians)

Ex: Show that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

$$\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)$$

$$\lim_{h\to 0} \frac{\cancel{1} - 2\sin^2\left(\frac{h}{2}\right) - \cancel{1}}{h} = \lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

Let $\theta = \frac{h}{2}$

$$= \lim_{\theta\to 0} \frac{-\cancel{2}\sin^2\theta}{\cancel{2}\theta} = -\lim_{\theta\to 0} \frac{\sin\theta \cdot \sin\theta}{\theta}$$

$$= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \sin\theta$$

$$= -(1)\cdot 0 = 0$$

Thm: $\boxed{\frac{d}{dx}\sin x = \cos x}$

Pf:

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x\,(0) + \cos x \cdot (1)$$

$$\frac{d}{dx}\sin x = \cos x.$$

Thm: $\boxed{\dfrac{d}{dx}\cos x = -\sin x}$

Pf: $\dfrac{d}{dx}\cos x = \dfrac{d}{dx}\sin\left(\dfrac{\pi}{2}-x\right) = \underbrace{\cos\left(\dfrac{\pi}{2}-x\right)\cdot(-1)}_{\text{chain Rule}} = -\sin x.$

Bude

Ex: Evaluate the derivatives of the following functions:

(a.) $f(x) = \sin(2x) - \cos(2x)$

(b.) $f(x) = x^2\cos(3x)$

(c.) $f(x) = \dfrac{\sin x}{1+\cos x}$

d) $f(x) = (x + \csc x)\cot x$

e) $f(x) = \dfrac{1+\sec x}{1-\sec x}$

f) $f(x) = 5x^2\sec x - x^3\tan x$

Soln:

(a) $f(x) = \sin(2x) - \cos(2x)$

$f'(x) = 2\cos 2x + 2\sin 2x = 2(\cos 2x + \sin 2x)$

(b) $f(x) = x^2\cos(3x)$

$f'(x) = 2x\cdot\cos(3x) + x^2\dfrac{d}{dx}\cos(3x)$

$= 2x\cos(3x) - 3x^2\sin(3x).$

(c) $f(x) = \dfrac{\sin x}{1+\cos x}$

$f'(x) = \dfrac{\cos x\,(1+\cos x) - (-\sin x)\sin x}{(1+\cos x)^2} = \dfrac{\cos x + \cos^2 x + \sin^2}{(1+\cos x)^2}$

$= \dfrac{1+\cos x}{(1+\cos x)^2} = \boxed{\dfrac{1}{1+\cos x}}$

Ex: Use two different methods to find the derivative of the function.

$$f(t) = \sin t \cos t.$$

I.st way:

$$f'(t) = \cos t \cos t + (-\sin t)\sin t = \underline{\cos^2 t - \sin^2 t}$$

2nd way:

$$\sin(2t) = 2 \sin t \cos t \quad //$$

$$f(t) = \frac{1}{2}\sin(2t)$$

$$f'(t) = \frac{1}{2} \cdot 2 \cos 2t = \underline{\cos 2t}$$

The Derivative of the Other Trigonometric Functions

1.) $\frac{d}{dx} \tan x = \sec^2 x$

2.) $\frac{d}{dx} \cot x = -\csc^2 x$

3.) $\frac{d}{dx} \csc x = -\csc x \cot x$

4.) $\frac{d}{dx} \sec x = \sec x \tan x$

Ex: Show that

(a.) $\frac{d}{dx} \cot x = -\csc^2 x$

$$\frac{d}{dx} \cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{(-\sin x)(\sin x) - \cos x \cos x}{(\sin x)^2}$$

$$= -\frac{\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = \underline{-\csc^2}$$

(6.) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Soln: $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$

$= -\frac{1}{\sin^2 x}\cos x$

$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$

$\frac{d}{dx}(\csc x) = -\cot x \csc x$

Ex: (a) $\frac{d}{dx}\left(\frac{3}{\sin(2x)}\right) = \frac{d}{dx}(3\csc(2x)) = (2)(3)(-\csc(2x)\cot(2x))$

$= -6\csc(2x)\cot(2x)$

(b.) $y = \sec\left(\frac{1}{x}\right)$

$y' = \frac{-1}{x^2}\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)$

Ex: Find the tangent and normal lines to the curve

$y = \tan\left(\frac{\pi x}{4}\right)$ at the point $(1,1)$.

Soln: $y'(x) = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}x\right)$.

$y'(1) = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{4}(\sqrt{2})^2 = \frac{\pi}{2}$. $\rightarrow$ slope of tangent line

Eqn of tangent line:

$y - 1 = \frac{\pi}{2}(x-1) \Rightarrow \boxed{y = \frac{\pi}{2}x - \frac{\pi}{2} + 1}$

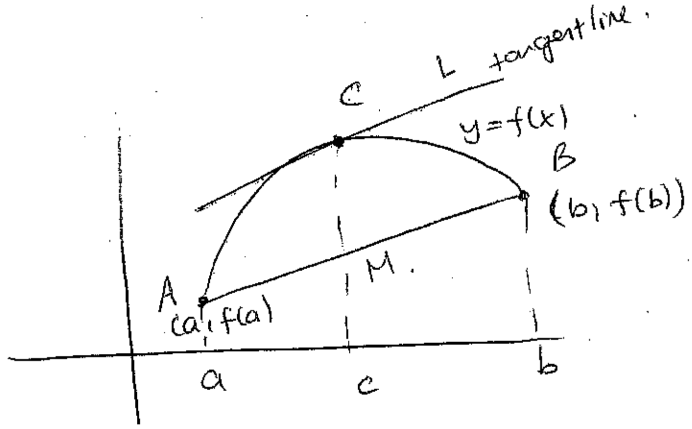
slope of normal $= -\dfrac{1}{\frac{\pi}{2}} = -\dfrac{2}{\pi}$

Eqn of normal line:

$$y - 1 = -\frac{2}{\pi}(x-1) \implies \boxed{y = -\frac{2}{\pi}x + \frac{2}{\pi} + 1}$$

Sec 2.6 The Mean-Value Thm:

Thm Mean-Value Thm:



Suppose f is continuous on the closed, finite interval $[a,b]$ and it is differentiable on open interval (a,b).

Then $\exists$ a point $c \in (a,b)$ s.t.

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

~~ie/ slope of L~~

ie/ slope of L = slope of M

$\Rightarrow$ L $\parallel$ M.

Ex Verify the conclusion of the Mean-Value Thm for $f(x) = \sqrt{x}$ on $[a,b]$ where $0 \le a < b$.

Soln: By mean value thm $\exists$ $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{b}-\sqrt{a}}{b-a} = \frac{(\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})}{(b-a)(\sqrt{b}+\sqrt{a})} = \frac{\cancel{b-a}}{(\cancel{b-a})(\sqrt{b}+\sqrt{a})} = \frac{1}{\sqrt{b}+\sqrt{a}}$$

$$\Rightarrow \quad 2\sqrt{c} = \sqrt{b}+\sqrt{a} \quad \Rightarrow \quad c = \left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2$$

$a < b$.

$$\Rightarrow \quad \underbrace{\left(\frac{\sqrt{a}+\sqrt{a}}{2}\right)^2}_{a} < \underbrace{\left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2}_{c} < \underbrace{\left(\frac{\sqrt{b}+\sqrt{b}}{2}\right)^2}_{b}$$

$$a < c < b$$

$\Rightarrow \quad c \in (a,b)$

Ex: Show that $\sin x < x$ for all $x > 0$.

Soln: if $x > 2\pi \quad \Rightarrow \quad \sin x \leq 1 < 2\pi < x$

$\Rightarrow \quad \sin x < x$

if $0 < x \leq 2\pi$

By Mean Value thm, $\exists\ c \in (0, 2\pi)$

s.t. $\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x - 0} = \left.\frac{d}{dx}\sin x\right|_{x=c} = \cos c < 1$.

$\Rightarrow \quad \sin x < x$.

Ex: Show that $\sqrt{1+x} < 1+\frac{x}{2}$ for $x>0$
and for $-1 \le x < 0$

Soln:
If $x>0$, Apply Mean Value thm to

$$f(x)=\sqrt{1+x} \quad \text{on} \quad [0,x]$$

$\Rightarrow \exists \ c \in (0,x)$ s.t.

$$\frac{\sqrt{1+x}-1}{x} = \frac{f(x)-f(0)}{x-0} = f'(c) = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$ because $c>0$.

$$\sqrt{1+x}-1 < \frac{1}{2}x \Rightarrow \sqrt{1+x} < \frac{1}{2}x+1$$

If $-1 \le x < 0$, apply mean value thm to

$$f(x)=\sqrt{1+x} \quad \text{on} \quad [x,0]$$

$\Rightarrow \exists \ c \in (x,0)$ s.t.

$$\frac{1-\sqrt{1+x}}{-x} = \frac{f(0)-f(x)}{0-x} = f'(c) = \frac{1}{2\sqrt{1+c}} > \frac{1}{2}$$

$-1 \le c < 0$
$0 \le c+1 < 1$

$$\frac{\sqrt{1+x}-1}{x} > \frac{1}{2} \quad \cancel{\Rightarrow \sqrt{1+x} > \frac{1}{2}x+1}$$

$$\sqrt{1+x}-1 < \frac{1}{2}x \quad \text{because } x<0$$

$$\sqrt{1+x} < \frac{1}{2}x+1$$

Increasing and Decreasing Functions

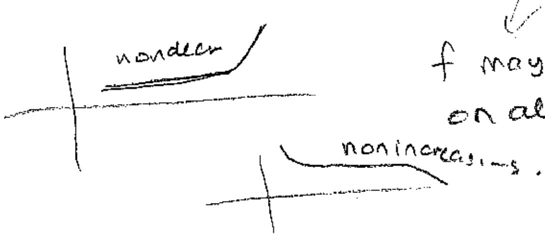
Defn: Suppose f is defined on an interval I, and x_1 and x_2 are two points of I.

(a.) If $f(x_2) > f(x_1)$ when $x_2 > x_1 \Rightarrow f$ is <u>increasing</u> on - (function take different value at different pon)

(b) if $f(x_2) < f(x_1)$ " $x_2 > x_1 \Rightarrow f$ is <u>decreasing</u> on

(c.) if $f(x_2) \geq f(x_1)$ " $x_2 > x_1 \Rightarrow f$ is <u>nondecreasin</u> on I

(d.) if $f(x_2) \leq f(x_1)$ " $x_2 > x_1 \Rightarrow f$ is <u>non increasin</u> on I



f may be constant on all or part of interv

Thm

Let J be an open interval and let I be an interval consisting of all the points in J and one or both endpoints of J.

Suppose f is continuous on I and differentiable on J.

(a.) if $f'(x) > 0 \quad \forall\, x \in J \Rightarrow f$ is increasing on I

(b.) if $f'(x) < 0 \quad \forall\, x \in J \Rightarrow f$ is decreasing on I

(c) if $f'(x) \geq 0$ " $\Rightarrow f$ is nondecreasing on I

(d.) if $f'(x) \leq 0$ " $\Rightarrow f$ is nonincreasing on I.

Ex: On what intervals is the ~~func~~

$$f(x) = x^3 - 12x + 1 \quad \text{increasing}$$

on what intervals is it decreasing?

Soln:

$$f'(x) = 3x^2 - 12 = 3(x^2-4) = 3(x-2)(x+2).$$

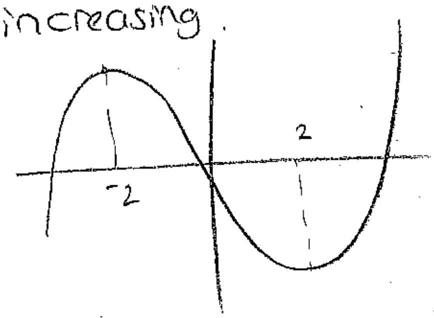
If $x > 2 \Rightarrow (x-2) > 0 \Rightarrow f'(x) > 0.$

or if $x < -2 \Rightarrow \underset{<0}{(x-2)}\underset{<0}{(x+2)} > 0 \Rightarrow f'(x) > 0.$

if $-2 < x < 2 \Rightarrow f'(x) < 0.$

So, on $(-\infty, -2)$ and $(2, \infty)$ f is increasing.

on $(-2, 2)$ f is decreasing.



Ex: Let $f(x) = (x^2-4)^2$

Soln:

$$f'(x) = 2(x^2-4) \cdot 2x$$

$$\boxed{f'(x) = 2(x-2)(x+2) \cdot 2x}$$

if $x < -2$ $\quad f'(x) = 2\underset{<0}{(x-2)}\underset{<0}{(x+2)} \cdot \underset{<0}{2x} < 0$

f is decreasing on $(-\infty, -2)$.

if $x > 2$ $\quad f'(x) = 2(x-2)(x+2) \cdot 2x > 0.$

f is increasing on $(2, \infty)$.

if $-2 < x < 0$ $\quad f'(x) = 2\underbrace{(x-2)}_{<0}\underbrace{(x+2)}_{>0}\cdot\underbrace{2x}_{<0} > 0$

f is increasing on $(-2, 0)$.

if $\quad 0 < x < 2 \quad f'(x) = 2\underbrace{(x-2)}_{<0}\underbrace{(x+2)}_{>0}\cdot\underbrace{2x}_{>0} < 0$

f is decreasing on $(0, 2)$.

<u>Thm:</u> If f is continuous on an interval I, and $f'(x) = 0$ at every interior point of I

$\Rightarrow$ $f(x) = C$, constant function on I.

<u>Thm:</u> If f is defined on an open interval (a, b) and ~~acti~~ has a maximum (or minimum) value at the point c in (a, b) and if $f'(c)$ exists $\Rightarrow$ $f''(c) = 0$

Values of x where $f'(x) = 0$ are called <u>critical points</u> of the function f.

Read

Thm: Rolle's Thm

Suppose that the function g is continuous on the closed finite interval $[a,b]$ and it is differentiable on the open interval (a,b).

If $g(a) = g(b)$, then there exists $c \in (a,b)$ s.t

$g'(c) = 0$.

Note: Rolle's thm is a special case of the Mean-Value thm. in which the chord line M and tangent line L have slope $= 0$.

Thm: Generalized Mean-Value Thm:

If functions f and g are both continuous on $[a,b]$ and differentiable on (a,b), and if $g'(x) \neq 0$ for every x in (a,b)

$\Rightarrow \exists \; c \in (a,b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Sec 2.7 Using Derivatives:

Approximating Small changes:

~~If~~ $y = f(x)$

then ~~To know how~~ a change in the value of x by an amount Δx will affect the value of y,

$$\boxed{\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x}$$

exact change in y.

$$\boxed{\text{Relative change in } x = \frac{\Delta x}{x}.}$$

measuring change in quantity wrt size of the quantity.

$$\boxed{\text{Percentage change in } x = 100\frac{\Delta x}{x}.}$$

Ex By approximately what percentage does the area of a circle increase if the radius increases by 2%?

Soln: Area $= A = \pi r^2$

$$\Delta A \approx \frac{dA}{dr}\Delta r = 2\pi r . \Delta r$$

$$\frac{\Delta A}{A} \approx \frac{2\cancel{\pi} r\,\Delta r}{\cancel{\pi} r^2} = 2\frac{\Delta r}{r}$$

If r increases by 2% $\Rightarrow \Delta r = \frac{2}{100} \cdot r \quad \Rightarrow \frac{\Delta r}{r} = \frac{2}{100}$

$$\frac{\Delta A}{A} \approx 2 \times \frac{2}{100} = \frac{4}{100}$$

Area increases by 4% .

Binder

Defn:

* The average rate of change of $f(x)$ wrt x over the interval from a to $a+h$ is,

$$\frac{f(a+h) - f(a)}{h}$$

* The (instantaneous) rate of change of f wrt x at $x=c$ is,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$, provided limit ex

Ex How fast is area of A circle increasing wrt its radius when the radius is 5 m ?

Soln:

The rate of change of A wrt r is

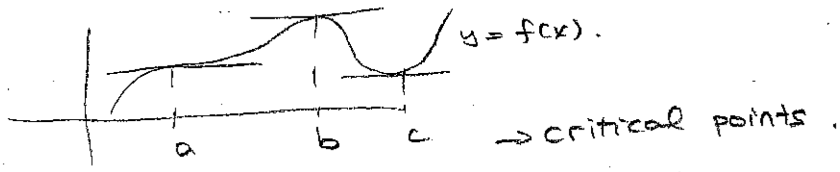
$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

when $r = 5$, area is changing at a rate $2\pi(5) = 10\pi$

This means small change Δr in radius when $r = 5$ this would result a change $10\pi \Delta r$ in area .

Defn: If $f'(x_0) = 0$ then f is stationary at x_0 and x_0 is called critical point of f

The graph has a horizontal tangent at a critical point and f may or may not have a max or min value there.



Ex: The temperature at a certain location t hours after noon on a certain day is T°C

$$T = \frac{1}{3}t^3 - 3t^2 + 8t + 10 \qquad 0 \le t \le 5.$$

How fast is the temperature rising or falling at 1:00pm at 3:00 pm? At what instants is the temperature stationary. ?

Soln: The rate of change of temp is.

$$\frac{dT}{dt} = t^2 - 6t + 8 = (t-2)(t-4).$$

$t = 1 \Rightarrow \frac{dT}{dt} = 8 - 5 = 3$ temp is rising at rate 3°C / hour at 1:00 pm.

$t = 3 \Rightarrow \frac{dT}{dt} = -1$ temp is falling at a rate of 1°C / hour at 3:00 pm.

Temperature is stationary when $\frac{dT}{dt} = (t-2)(t-4) = 0$.
at 2:00 pm and 4:00 pm.

Sec 2.8 Higher - Order Derivatives :

If $y' = f'(x)$ of function $y = f(x)$ is itself differentiable at x, we can calculate its derivative, It is called the second derivative of f, $y'' = f''(x)$.

$$\boxed{y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = D_x^2\, y = D_x^2 f(x)}$$

Similarly we can consider, in general n th-order derivative of a function.

We show the higher order derivatives as;

$$\boxed{y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}f(x) = D_x^n\, y = D_x^n f(x)}$$

Ex: 1 Let $y = x^4$

$y' = 4x^3$

$y'' = 12x^2$

$y''' = 24x$ $\qquad y^{(5)} = 0$

$y^{(4)} = 24$ and all higher derivatives are zero.

In general,

$$f(x) = x^n$$

$$f^{(k)}(x) = n(n-1)(n-2) \cdots (n-(k-1))\, x^{n-k}$$

$$= \begin{cases} \dfrac{n!}{(n-k)!}\, x^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n. \end{cases}$$

$n! \rightarrow$ n factorial.

$0! = 1$

$1! = 0! \times 1 = 1 \times 1 = 1$

$2! = 1! \times 2 = 1 \times 2 = 2$

$3! = 2! \times 3 = 2 \times 3 = 6$

$\vdots$

$n! = (n-1)! \times n = 1 \times 2 \times \cdots (n-1) \times n$.

For a polynomial P of degree n,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$P^{(n)}(x) = n!\, a_n$$

<u>Ex 2</u> Find the n'th derivative, $f^{(n)}(x)$ of

$$f(x) = \frac{1}{2-x} = (2-x)^{-1}$$

<u>Soln:</u>

$$f'(x) = -(2-x)^{-2}\cdot(-1) = (2-x)^{-2}$$

$$f''(x) = -2(2-x)^{-3}\cdot(-1) = 2(2-x)^{-3}$$

$$f'''(x) = -6(2-x)^{-4}\cdot(-1) = 6(2-x)^{-4}$$

$$f^{(4)}(x) = -24(2-x)^{-5}(-1) = 24(2-x)^{-5}$$

$$f^{(n)}(x) = n!\,(2-x)^{-(n+1)}$$

To prove it is true for all $n \geq 1$ use mathematical induction.

Assume it is true for $n=k$. k: post integer.

try to prove it is true for $n=k+1$

$$f^{(k+1)}(x) = \frac{d}{dx}\left(f^{k}\right)(x) = \frac{d}{dx}\left[k!\,(2-x)^{-(k+1)} \right]$$

$$= -k!\,(k+1)\,(2-x)^{-(k+1)-1}\,(-1)$$

$$= +(k+1)!\,(2-x)^{-k-1-1}$$

$$f^{(k+1)}(x) = (k+1)!\,(2-x)^{-(k+2)}$$

<u>Ex3</u> Find a formula for $f^{(n)}(x)$

if $f(x) = \cos(ax)$

<u>Soln:</u>

$$f'(x) = -a\sin(ax)$$

$$f''(x) = -a^2\cos(ax)$$

$$f'''(x) = a^3\sin(ax).$$

$$f^{IV}(x) = a^4\cos(ax)$$

$$f^{(5)}(x) = -a^5\sin(ax).$$

$$f^{n}(x) = \begin{cases} (-1)^{k}\,a^{n}\cos ax & n=2k. \\ (-1)^{k+1}\,a^{n}\sin(ax). & n=2k+1 \end{cases}$$

Ex 4 show that $y = \tan kx$ is a soln of

$$\frac{d^2y}{dx^2} - 2k^2 y(1+y^2) = 0.$$

Soln:

$$\frac{dy}{dx} = k.\sec^2 kx = k.(\sec kx)^2$$

$$\frac{d^2y}{dx^2} = 2k.k(\sec kx).\sec kx \tan kx$$

$$= 2k^2 \sec^2 kx \tan kx$$

$\sec^2 kx = 1 + \tan^2 k$

$$= 2k^2 \tan kx (1 + \tan^2 kx)$$

$$= 2k^2 y (1+y^2)$$

$$\frac{d^2y}{dx} - 2k^2 y(1+y^2) = 0.$$

2.9 Implicit Differentiation :

(If curve is graph of function $y = f(x)$ $\Rightarrow$ we can find its slope by finding f'.

Not all curves are graph of such functions.)

Curves are generally graphs of eqns in two variable,

$$F(x,y) = 0.$$

Ex $x^2 + y^2 - 25 = 0$ $\rightarrow$ ~~unit circle~~ with centre at origin. with radius 5.

We can find the derivative $\frac{dy}{dx}$ of these functions by method implicit differentiation

To find $\frac{dy}{dx}$ by implicit differentiation :

1.) Differentiate both sides of the eqn wrt x, regarding y as a function of x and use Chain Rule to differentiate functions of y.

2.) Collect terms with $\frac{dy}{dx}$ on one side of the eqn and solve for $\frac{dy}{dx}$ by dividing by its coefficient.

Ex 1 Find $\frac{dy}{dx}$ if $y^2 = x$

Soln:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

$$2y \cdot \frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Ex 2 Find $\frac{dy}{dx}$ in terms of x and y.

$$x^2 y^3 = 2x - y$$

$$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(2x) - \frac{d}{dx}(y)$$

$$2x \cdot y^3 + x^2 \frac{d}{dx}(y^3) = 2 - \frac{dy}{dx}$$

$$2xy^3 + x^2 \cdot 3y^2 \cdot \frac{dy}{dx} = 2 - \frac{dy}{dx}$$

$$3x^2y^2 \frac{dy}{dx} + \frac{dy}{dx} = 2 - 2xy^3$$

$$\frac{dy}{dx}(3x^2y^2+1) = 2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{2-2xy^3}{3x^2y^2+1}$$

<u>Ex 3</u> Find an eqn of the tangent to the ~~given~~ curve

$x^2y^3 - x^3y^2 = 12$ at $(-1, 2)$.

<u>Soln:</u> To find slope of tangent line differentiate curve wrt x

$$\frac{d}{dx}(x^2y^3) - \frac{d}{dx}(x^3y^2) = \frac{d}{dx}(12)$$

$$2xy^3 + x^2 \cdot 3y^2 \cdot \frac{dy}{dx} - \left[3x^2 \cdot y^2 + x^3 \cdot 2y\frac{dy}{dx}\right] = 0.$$

$$2xy^3 + 3x^2y^2\frac{dy}{dx} - 3x^2y^2 - 2x^3y\frac{dy}{dx} = 0.$$

Subst $x = -1$, $y = 2$.

$$-16 + 12\frac{dy}{dx} - 12 + 4\frac{dy}{dx} = 0$$

$$16\frac{dy}{dx} = 28 \implies \frac{dy}{dx} = \frac{28}{16} = \frac{7}{4}.$$

slope of tangent line at $(-1,2)$ is $\frac{7}{4}$.

Eqn of tangent line,

$$y - 2 = \frac{7}{4}(x+1) \Rightarrow \boxed{y = \frac{7}{4}x + \frac{15}{4}}$$

Buda'

x Ex:4 Show that for any constants a and b the curves $x^2 - y^2 = a$ and $xy = b$ intersect at right angles, that is, at any point where they intersect their tangents are perpendicular.

Soln:

slope at any point on $x^2 - y^2 = a$ is

$$2x - 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

slope at any point on $xy = b$ is

$$1 \cdot y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

If two curves intersect at (x_0, y_0)

$\Rightarrow$ their slopes at that point are:

$$\frac{x_0}{y_0} \quad \text{and} \quad \frac{-y_0}{x_0}$$

$\Rightarrow$ tangent line to one curve is normal line to other curve at that point.

$\Rightarrow$ curves intersect at right angles.

Ex 5 Find y'' in terms of x and y.

$$xy = x + y$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x) + \frac{d}{dx}(y)$$

$$1 \cdot y + x\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$x\frac{dy}{dx} - \frac{dy}{dx} = 1 - y$$

$$y' = \frac{dy}{dx} = \frac{1-y}{x-1}$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-y}{x-1}\right)$$

$$y'' = \frac{\left[\frac{d}{dx}(1-y)\right](x-1) - \left[\frac{d}{dx}(x-1)\right](1-y)}{(x-1)^2}$$

$$= \frac{-\frac{dy}{dx}(x-1) - (1-y)}{(x-1)^2}$$

$$y'' = \frac{-\left(\frac{1-y}{\cancel{x-1}}\right)\cdot(\cancel{x-1}) - (1-y)}{(x-1)^2} = \frac{-2(1-y)}{(x-1)^2}$$

$$y'' = \frac{2(y-1)}{(x-1)^2}$$

//

Sec 2-10 Antiderivatives & Initial-Value Problems

Defn: An antiderivative of a function f on an interval I is another function F satisfying

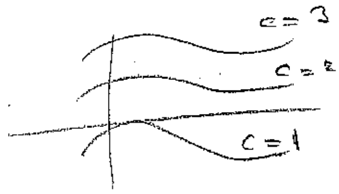
$$F'(x) = f(x) \quad \text{for } x \text{ in } I$$

Ex1 a) $F(x) = x$ is antiderivative of $f(x) = 1$ on any interval,

$F'(x) = 1 = f(x)$.

(b.) $F(x) = -\frac{1}{x}$ is antiderivative of $f(x) = \frac{1}{x^2}$ on any interval not containing $x = 0$,

$$F'(x) = \frac{1}{x^2} = f(x) \quad \text{everywhere except } x = 0.$$



Note: Antiderivatives are not unique.

Ex $F(x) = x + c$, c = constant is antiderivative of $f(x) = 1$

Defn: The indefinite integral of $f(x)$ on interval I is,

$$\boxed{\int f(x)\,dx = F(x) + C} \quad \text{on } I$$

where $F'(x) = f(x)$ for all x in I.

Ex: (a.) $\int x\,dx = \frac{x^2}{2} + C$

(b.) $\int (2x^3 + 5x^2 + 4)\,dx = 2 \cdot \frac{x^4}{4} + \frac{5x^3}{3} + 4x + C$

Antiderivatives of some functions :

(1) $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1.$

Ex: a.) $\int 1\,dx = \int x^0\,dx = x + C.$

b.) $\int x\,dx = \frac{x^2}{2} + C$

c.) $\int x^2\,dx = \frac{x^3}{3} + C$

d.) $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

e.) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C.$

(2.) $\int \sin x\,dx = -\cos x + C$

(3.) $\int \cos x\,dx = \sin x + C$

(4.) $\int \sec^2 x\,dx = \tan x + C$

(5.) $\int \csc^2 x\,dx = -\cot x + C$

(6.) $\int \sec x \tan x\,dx = \sec x + C$

(7) $\int \csc x \cot x\,dx = -\csc x + C$

Differential Equations and Initial-Value Problems :

<u>Ex:</u> show that for any constants A, B, $y = Ax^3 + \frac{B}{x}$ is a solution of the diff eqn

$$x^2 y'' - x y' - 3y = 0$$ on any interval not containing 0.

<u>Soln:</u>

$$y' = 3Ax^2 + Bx^{-2}$$

$$y'' = 6Ax + 2Bx^{-3} = 6Ax + \frac{2B}{x^3}$$

$$x^2 \cdot \left(6Ax + \frac{2B}{x^3}\right) - x\left(3Ax^2 - \frac{B}{x^2}\right) - 3y = 0$$

$$6Ax^3 + \frac{2B}{x} - 3Ax^3 + \frac{B}{x} - 3y = 0$$

$$y = Ax^3 + \frac{B}{x}$$

$$6\cancel{Ax^3} + \frac{2B}{x} - 3\cancel{Ax^3} + \frac{B}{x} - 3\cancel{Ax^3} - 3\frac{B}{x} = 0.$$

<u>Defn:</u> An <u>initial-value problem</u> (IVP) consists of :

(1-) a differential eqn to be solved

(2-) prescribed values for the soln and enough of its derivatives at a particular point (initial point) to determine values for all the arbitrary constants in general soln of the DE.

Ex: Find the function $f(x)$ whose derivative is

$$f'(x) = 2x^3 + x \quad \text{for all real } x, \text{ where } f(1) = 3$$

Soln:

$$f(x) = \int (2x^3 + x)\,dx = \frac{2x^4}{4} + \frac{x^2}{2} + C = \frac{1}{2}x^4 + \frac{1}{2}x^2 + C$$

$$f(1) = \frac{1}{2}(1)^4 + \frac{1}{2}(1)^2 + C = 3 \quad \Rightarrow \quad 1 + C = 3$$

$$\Rightarrow \quad \boxed{C = 2}$$

$$f(x) = \frac{1}{2}x^4 + \frac{1}{2}x^2 + 2 \quad //$$

Ex: Find the function $g(t)$ whose derivative is $\frac{6x-6}{x^{4/3}}$ and whose graph passes through $(1, 2)$

Soln:

$$g(t) = \int \frac{6x-6}{x^{4/3}}\,dx = \int (6x^{-1/3} - 6x^{-4/3})\,dx$$

$$= \frac{6x^{2/3}}{\frac{2}{3}} - \frac{6x^{-1/3}}{-1/3} + C = 9x^{2/3} + 18 \cdot x^{-1/3} + C$$

$$g(1) = 9(1)^{2/3} + 18(1)^{-1/3} + C = 2$$

$$27 + C = 2 \quad \Rightarrow \quad \boxed{C = -25}$$

$$g(t) = 9x^{2/3} + 18x^{-1/3} + 25. \quad //$$

Ex: Use the previous example, to solve the following initial-value problem.

$$x^2y'' - xy' - 3y = 0 \qquad (x>0)$$

$$y(1) = 2$$

$$y'(1) = -6$$

Soln:

$$y = Ax^3 + \frac{B}{x} \quad \text{is soln of diff eqn.}$$

$$y' = 3Ax^2 - \frac{B}{x^2}$$

$$x=1 \Rightarrow y=2 \qquad 2 = A(1)^3 + \frac{B}{1} \Rightarrow \quad 2 = A+B$$

$$x=1 \Rightarrow y'=-6 \qquad -6 = 3A - B$$

$$\begin{array}{r} 2 = A + B \\ + \; -6 = 3A - B \\ \hline -4 = 4A \end{array}$$

$$\boxed{\begin{array}{c} A=-1 \\ B=3 \end{array}}$$

$$\therefore \quad y = -x^3 + \frac{3}{x} \quad \text{for } x>0$$

soln of IVP.

Ex: Solve the initial-value problem

$$\begin{cases} y' = 3\sqrt{x} \\ y(4) = 1 \end{cases}$$

where is the soln valid?

Soln:

$$y = \int 3\sqrt{x}\,dx = \int 3x^{1/2}\,dx = \cancel{3}\,\frac{2}{\cancel{3}}x^{3/2} + C = 2x^{3/2} + C$$

$$y(4) = 2(\sqrt{4})^3 + C = 1 \Rightarrow 16 + C = 1 \Rightarrow \underline{\underline{C = -15}}$$

$$y = 2\left(\sqrt{x}\right)^{3} - 15 = 2x^{2/3} - 15$$

it is valid for $\underline{\underline{x > 0}}$.

<u>Ex:</u> Solve the second-order IVP

$$\begin{cases} y'' = x^3 - 1 \\ y(0) = 0 \\ y'(0) = 8 \end{cases}$$

On what interval is the soln valid?

<u>Soln:</u> $y'' = (y')'$

$$y'(x) = \int x^3 - 1 \, dx = \frac{x^4}{4} - x + C_1$$

$$y'(0) = C_1 = 8$$

$$y'(x) = \frac{x^4}{4} - x + 8$$

$$y(x) = \int \frac{x^4}{4} - x^{1} \overset{8}{} \, dx = \frac{x^5}{20} - \frac{x^2}{2} + 8x + C_2$$

$$y(0) = C_2 = 0$$

$$y(x) = \frac{x^5}{20} - \frac{x^2}{2} + 8x$$ is valid for all x.

(kitaptaki)

Ex: Solve the initial-value problem

$$\begin{cases} y' = \dfrac{3+2x^2}{x^2} \\ y(-2) = 1 \end{cases}$$

Where is the soln valid ?

Soln:

$$y(x) = \int (3x^{-2} + 2)\,dx = \frac{3x^{-1}}{-1} + 2x + C$$

$$= -\frac{3}{x} + 2x + C$$

$$y(-2) = \frac{-3}{-2} + 2(-2) + C = 1$$

$$\frac{3}{2} - 4 + C = 1 \Rightarrow C = \frac{7}{2}$$

$$y = \frac{-3}{x} + 2x + \frac{7}{2} \quad //.$$

it is only soln of given IVP for $x < 0$.

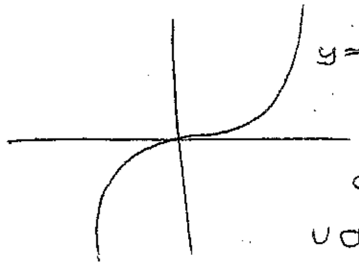
Because $(-\infty, 0)$ is the largest interval which contains -2 but not $x = 0$ where DE is undefined.

Chapter 3 Transcendental Functions :

functions that can not be constructed.
Ex: Trigonometric funct.

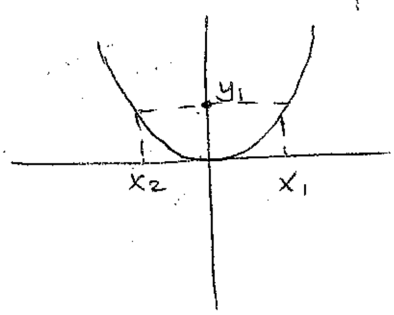
3.1 Inverse Functions :

Consider $f(x) = x^3$



different values of x give different values of $f(x)$

Consider $f(x) = x^2$



Defn: A function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$ or equivalently

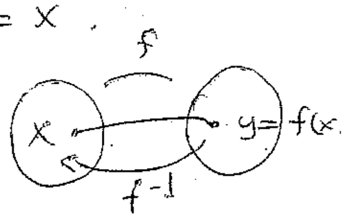
if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

* A function defined on a single interval is one-to-one there if it is either increasing or decreasing

Ex: $f(x) = x^3$ is a one-to-one function.

Defn: If f is one-to-one then it has an inverse function f^{-1}. The value of $f^{-1}(x)$ is the unique number y in the domain of f for which $f(y) = x$.

Thus, $y = f^{-1}(x) \iff x = f(y)$.



Ex: Show that $f(x) = x^5 - 1$ is one to one function and find its inverse, $f^{-1}(x)$.

Soln: $f'(x) = 5x^4 > 0$ for every $x \in \mathbb{R}$.

f is increasing so f is one-to-one. $\Rightarrow$ it has inverse.

To find f^{-1}:

1-) $y = x^5 - 1$.

2.) interchange x and y

$$x = y^5 - 1$$

3.) Solve for y:

$$x + 1 = y^5$$

$$(x+1)^{1/5} = y.$$

$f^{-1}(x) = y \Leftrightarrow x = f(y)$.

4.) $f^{-1}(x) = (x+1)^{1/5}$

To check:

$(f \circ f^{-1})(x) = f[(x+1)^{1/5}] = ((x+1)^{1/5})^5 - 1 = x + 1 - 1 = x$ for all x in the domain of f^{-1}

$(f^{-1} \circ f)(x) = f^{-1}[x^5 - 1] = (x^5 - 1 + 1)^{1/5} = x$ for all x in the domain of f.

Properties of Inverse Functions:

1.) $y = f^{-1}(x) \iff x = f(y)$.

2.) The domain of f^{-1} is the range of f

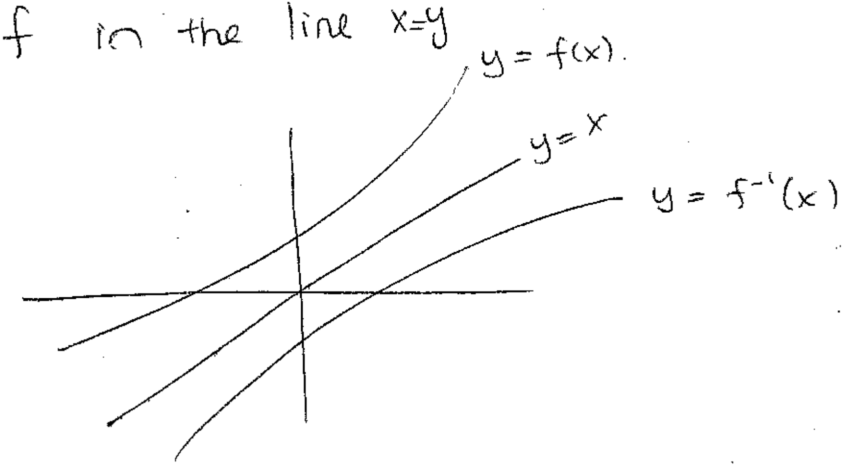
3.) The range of f^{-1} is the domain of f

✓4.) $f^{-1}(f(x)) = x$ for all x in the domain of f

✓5.) $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

6.) $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f

7.) The graph of f^{-1} is the reflection of the graph of f in the line $x=y$



Ex: Show that $g(x) = \sqrt{2x+1}$ is invertible and find its inverse.

Soln: if $\boxed{g(x_1) = g(x_2)} \Rightarrow \left(\sqrt{2x_1+1}\right)^2 = \left(\sqrt{2x_2+1}\right)^2$

$$2x_1 + \cancel{1} = 2x_2 + \cancel{1}$$

$$\boxed{x_1 = x_2.}$$

So g is one-to-one and invertible.

To find inverse :

1.) $y = \sqrt{2x+1}$

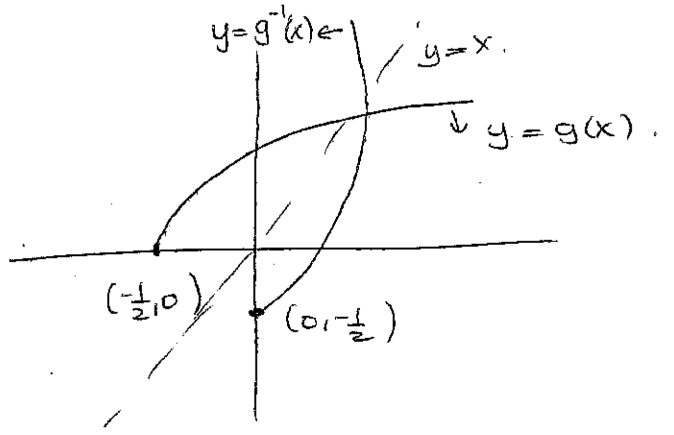
2.) $x = \sqrt{2y+1}$

3.) $x^2 = 2y+1 \Rightarrow 2y = x^2 - 1$

$y = \dfrac{x^2-1}{2}$

4.) $g^{-1}(x) = \dfrac{x^2-1}{2}$ for $x \geq 0$ ~~because range of g is $[0,\infty)$~~

By property 2) domain of $g^{-1}(x)$ = Range of $g(x)$ = $[0, \infty)$.



Defn: A function f is self-inverse if $\boxed{f^{-1} = f}$ that is $\boxed{f(f(x)) = x}$ for $\forall x$ in the domain of f.

Ex: $f(x) = \dfrac{1}{x}$ is self-inverse.

1) $y = \dfrac{1}{x}$

2) $x = \dfrac{1}{y}$

3) $y = \dfrac{1}{x}$.

$f(f(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{\frac{1}{x}} = x$.

Inverting Non One-to-One Functions :

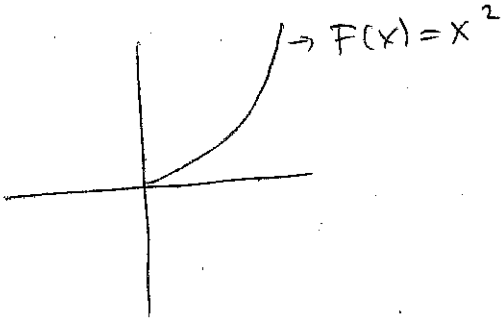
<u>Ex:</u> Trigonometric functions

$y = x^2$] not one-to-one functions.

For those functions we have to restrict the domain of the function so that the restricted function is one-to-one.

<u>Ex</u> Consider $F(x) = x^2$ for $0 \le x < \infty$.

F is invertible. (because one-to-one



1) $y = x^2$

2) $x = y^2$

3) $\sqrt{x} = y \Rightarrow F^{-1}(x) = \sqrt{x}$.

Derivatives of Inverse Functions :

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

Ex: $f(x) = x^2 - 4x + 5$, $x \geq 2$

Ex: $f(x) = \dfrac{3x+2}{2x-5}$

Ex: $f(x) = \dfrac{4x}{x-2}$

Ex: $f(x) = \sqrt[3]{5x+12}$

<u>Ex 2</u> Show that $f(x) = x^3 + x$ is one-to-one on the whole real line and $f(2) = 10$, find $(f^{-1})'(10)$.

Soln: $f'(x) = 3x^2+1 > 0$ for all real numbers x,

$\Rightarrow$ f is increasing $\Rightarrow$ f is 1-to-1 and invertible.

If $y = f^{-1}(x)$. we can't find f^{-1}.

$\Rightarrow$ $x = f(y) = y^3 + y$

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{d(x)}{dx} = \frac{d}{dx}(y^3+y).$$

$f'(x) = 3x^2+1$.

Let $y = f^{-1}(x) \Rightarrow f(y) = x$

$$1 = (3y^2+1)\cdot y'$$

$$= \frac{1}{f'(y)} = \frac{1}{3y^2+1}$$

$$(f^{-1}(x))' = y' = \frac{1}{3y^2+1}.$$

$$(f^{-1})'(10) = \left.\frac{1}{3y^2+1}\right|_{y=2}.$$

If $x = f(2) = 10$ $\Rightarrow$ $f^{-1}(10) = 2 = y$.

$$(f^{-1})'(10) = \left.\frac{1}{3y^2+1}\right|_{y=2} = \frac{1}{13}.$$

Budar

Ex: 3 13.) f is one-to-one with inverse f^{-1}

P.178/ Calculate the inverse of the given function in terms of f^{-1}

$$g(x) = f(x) - 2.$$

Soln: Let $y = g(x) = f(x) - 2$.

$$x = f(y) - 2$$

$$x+2 = f(y)$$

$$f^{-1}(x+2) = y \Rightarrow g^{-1}(x) = f^{-1}(x+2).$$

Ex: 4

P.175/ 3.) Show that

$$f(x) = \sqrt{x-1}$$

is one-to-one and calculate the inverse function f^{-1}.

Specify the domain and range.

Soln: if $f(x_1) = f(x_2) \Rightarrow \sqrt{x_1 - 1} = \sqrt{x_2 - 1}$

$$x_1 - 1 = x_2 - 1$$

$$x_1 = x_2.$$

So f is one-to-one.

$$y = \sqrt{x-1}$$

$$x = \sqrt{y-1} \Rightarrow x^2 = y-1$$

$$x^2 + 1 = y$$

$$f^{-1}(x) = x^2 + 1$$

$$D(f^{-1}) = R(f) = [0, \infty).$$

$$R(f^{-1}) = D(f) = [1, \infty).$$

Ex P.178/

(25.) Find $(f^{-1})'(x)$ if $f(x) = 1 + 2x^3$.

Soln $\frac{d}{dx}(f^{-1})(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$$f'(x) = 6x^2.$$

Ex: $(f^{-1}(x))'$ if $f(x) = x^3 + 2x - 1$

$y = 1+2x^3$

$x = 1+2y^3$

$\frac{x-1}{2} = y^3 \Rightarrow \left(\frac{x-1}{2}\right)^{1/3} = y \qquad f^{-1}(x) = \left(\frac{x-1}{2}\right)^{1/3}.$

$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(\left(\frac{x-1}{2}\right)^{1/3}\right)} = \frac{1}{6\left[\left(\frac{x-1}{2}\right)^{1/3}\right]^2} = \frac{1}{6\,(f^{-1}(x)}$$

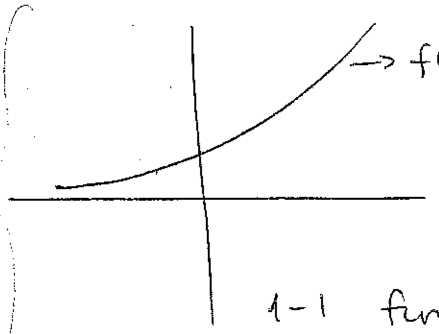
$$= \frac{1}{6\left(\frac{x-1}{2}\right)^{2/3}} =$$

Sec 3.2 Exponential & Logarithmic Functions:

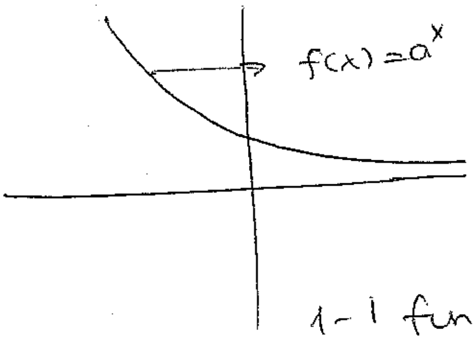
$f(x) = a^x$ is called exponential function.

base a is post. constant, , exponent x is variable.

If $a > 1$ if $0 < a < 1$



1-1 funct
increasing



1-1 function
decreasing.