$$y = 1+2x^3$$

$$x = 1+2y^3$$

$$\frac{x-1}{2} = y^3 \implies \left(\frac{x-1}{2}\right)^{1/3} = y \qquad f^{-1}(x) = \left(\frac{x-1}{2}\right)^{1/3}.$$

$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(\left(\frac{x-1}{2}\right)^{1/3}\right)} = \frac{1}{6\left[\left(\frac{x-1}{2}\right)^{1/3}\right]^2} = \frac{1}{6\left(f^{-1}(x)\right.}$$

$$= \frac{1}{6\left(\frac{x-1}{2}\right)^{2/3}} =$$

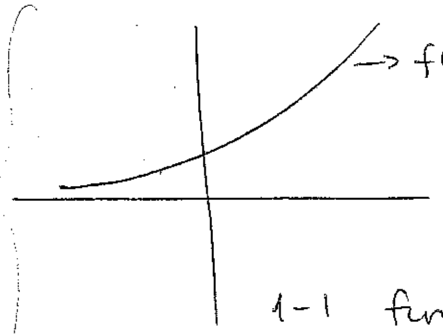
Sec 3.2 Exponential & Logarithmic Functions:

$f(x) = a^x$ is called exponential function.

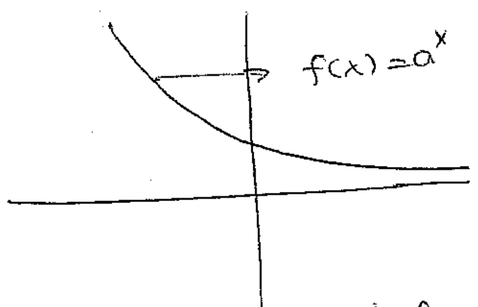
base a is post. constant, , exponent x is variable.

If $a > 1$ | if $0 < a < 1$

Not here



1-1 funct
increasing



1-1 function
decreasing.

Defn: Exponential Functions.

If $a > 0$

$\Rightarrow$ $a^0 = 1$

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}} \quad \text{if } n = 1, 2, 3, \ldots$$

$$a^{-n} = \frac{1}{a^n} \quad \text{if } n = 1, 2, 3, \ldots$$

$$a^{m/n} = \sqrt[n]{a^m} \quad \text{if } n = 1, 2, 3, \ldots \quad m = \mp 1, \mp 2, \mp 3, \ldots$$

How should we define a^x if x is not rational ?

Ex: 2^{π}

define, $a^x = \lim\limits_{\substack{r \to x}} a^r$ $\quad x$: irrational.

r = rational.

Laws of Exponents:

If $a > 0$ and $b > 0$ and x and y are any real numbers,

1.) $a^0 = 1$

2) $a^{x+y} = a^x \cdot a^y$.

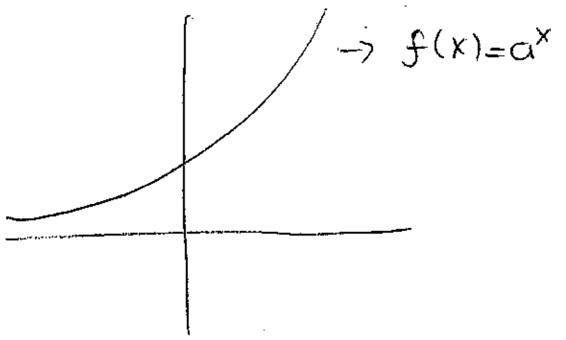
3.) $a^{-x} = \dfrac{1}{a^x}$

4.) $a^{x-y} = \dfrac{a^x}{a^y}$

5.) $(a^x)^y = a^{xy}$

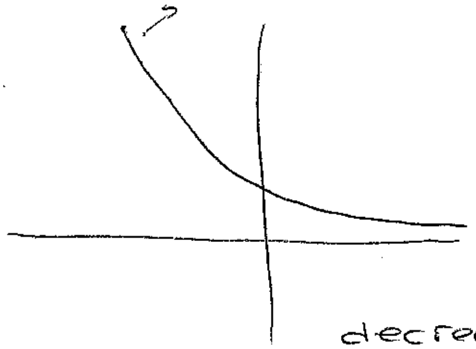
6.) $(ab)^x = a^x b^x$.

If $a>1$



increasing

$f(x)=a^x$ if $0<a<1$



decreasing.

If $a>1 \Rightarrow \lim_{x\to -\infty} a^x = 0 \qquad \lim_{x\to\infty} a^x = \infty$

if $0<a<1 \Rightarrow \lim_{x\to -\infty} a^x = \infty \qquad \lim_{x\to\infty} a^x = 0$.

If $a \neq 1$ the x-axis is horizontal asymptote for the graph of $y=a^x$.

Logarithms:

$y=a^x$ is 1-to-1 if $a>0$ and $a\neq 1$

so it has inverse which we call ~~this inverse function~~ a Logarithmic function.

Defn:

If $a>0$ and $a \neq 1$,

the function $\log_a x$ called the Logarithm of x to the base a. It is the inverse of the function a^x:

$$\boxed{y=\log_a x \iff x=a^y} \qquad (a>0,\ a\neq 1).$$

Domain of $\log_a x$: $(0, \infty)$ = Range of $x = a^y$
↓ takes only positive val.

Range of $\log_a x$ = Domain of $x = a^y$.
↳ defined for all real value y.

Range of $\log_a x$ = ~~all~~ $\mathbb{R}$.

$\boxed{\log_a (a^x) = x}$ for all real numb x

$\boxed{a^{\log_a x} = x}$ for all $x > 0$.

<u>Laws of Logarithms:</u>

If $a > 0$, $b > 0$, $a \neq 1$ and $b \neq 1$

$\Rightarrow$

(1.) $\log_a 1 = 0$

(2.) $\log_a (xy) = \log_a x + \log_a y$

(3.) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

(4.) $\log_a \left(\frac{1}{x}\right) = \log_a 1 - \log_a x = -\log_a x$.

(5.) $\log_a (x^y) = y \log_a x$

(6.) $\log_a x = \dfrac{\log_b x}{\log_b a}$

Ex: Simplify

(a-) $\log_2 10 + \log_2 12 - \log_2 15$

$= \log_2 (120) - \log_2 15 = \log_2\left(\frac{120}{15}\right) = \log_2 8 = \log_2 2^3 =$

$3\log_2 2 = 3$ //.

(b-) $\log_{a^2} a^3 = 3\log_{a^2} a = \frac{3}{2}\underbrace{\log_{a^2} a^2}_{=1} = \frac{3}{2}$

(c-) $3^{\log_9 4} =$

(c-) $2^{\log_4 8} = 2^{\frac{\log_2 8}{\log_2 4}} = 2^{\frac{3\log_2 2}{2\log_2 2}} = 2^{3/2}$.

d-) $3^{\log_9 4} = 3^{\frac{\log_3 4}{\log_3 9}} = \left(3^{\log_3 4}\right)^{\frac{1}{\log_3 3^2}} = (4)^{\frac{1}{2\log_3 3}} = 4^{1/2}$ //.

Ex: Solve the eqn.

$$3^{x-1} = 2^x.$$

Soln:

$\log_a 3^{x-1} = \log_a 2^x$

$(x-1)\log_a 3 = x\log_a 2$

$x\log_a 3 - \log_a 3 = x\log_a 2$

$(\log_a 3 - \log_a 2)\,x = \log_a 3$

$x = \dfrac{\log_a 3}{\log_a 3 - \log_a 2} = \dfrac{\log_a 3}{\log_a (3/2)}$

Ex: P.183/ 29.) Solve $\log_4(x+4) - 2\log_4(x+1) = \frac{1}{2}$ for x.

Soln: $\log_4(x+4) - \log_4(x+1)^2 = \frac{1}{2}$

$$\log_4 \frac{(x+4)}{(x+1)^2} = \frac{1}{2}$$

$$\log_4 \frac{(x+4)}{(x+1)^2} = \frac{1}{2}\log_4 4 = \log_4 4^{1/2}$$

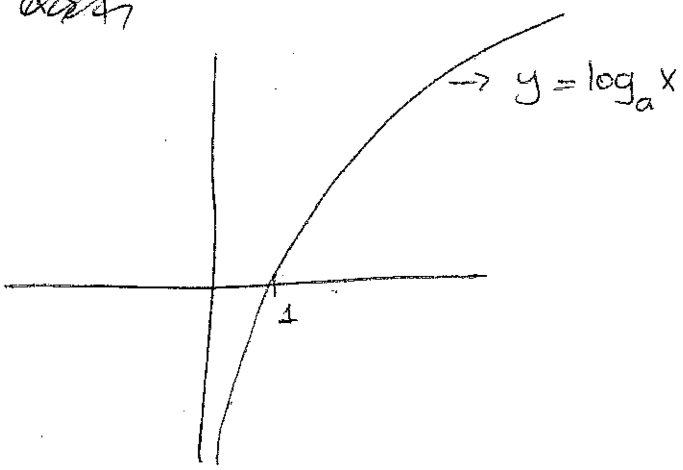
$$\frac{x+4}{(x+1)^2} = 2$$

$$x+4 = 2\left[x^2+2x+1\right] = 2x^2+4x+2$$

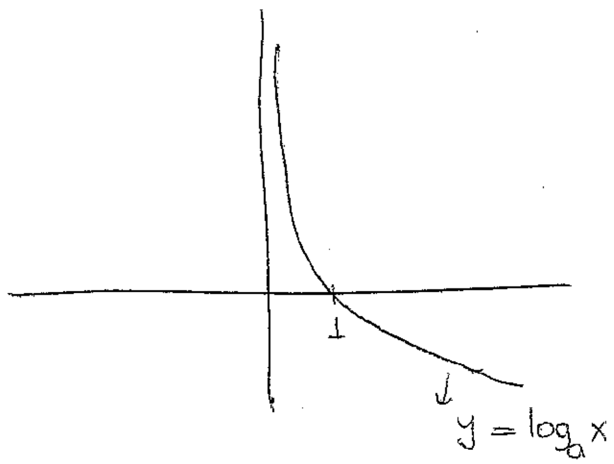
$$0 = 2x^2+3x-2$$

$$x_{1,2} = \frac{-3 \mp \sqrt{9-4(2)(-2)}}{4} = \frac{-3\mp 5}{4} \begin{cases} \frac{-8}{4} = -2 \\ \frac{2}{4} = \frac{1}{2} \end{cases}$$

$$\boxed{x=-2, \quad x=\frac{1}{2}}$$



$a>1$



$0<a<1$

if $a > 1 \Rightarrow \lim_{x \to 0^+} \log_a x = -\infty$ and $\lim_{x \to \infty} \log_a x = \infty$

if $0 < a < 1 \Rightarrow \lim_{x \to 0^+} \log_a x = \infty$ and $\lim_{x \to \infty} \log_a x = -\infty$.

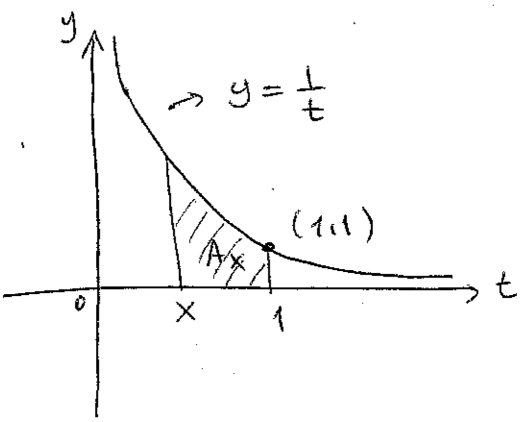
Sec 3.3 The Natural Logarithm and Exponential:

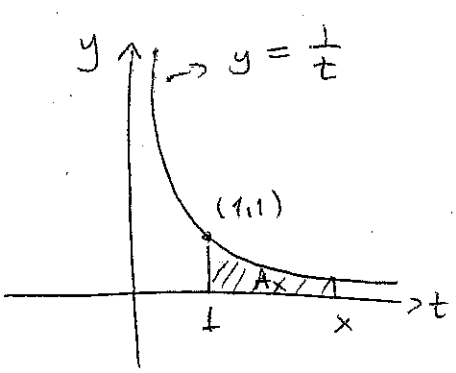
Defn: The Natural Logarithm

For $x>0$, let A_x be the area of the plane region bounded by the curve $y=\frac{1}{t}$ and t-axis and the vertical lines $t=1$ and $t=x$.

Then, define,

$$\ln x = \begin{cases} A_x & \text{if } x \geq 1 \\ -A_x & \text{if } 0 < x < 1 \end{cases}$$





Note: * $\boxed{\ln 1 = 0}$

* $\ln x > 0$ if $x \geq 1$

  $\ln x < 0$ if $0 < x < 1$

* $\ln x$ is a 1-to-1 function.

Graph of $\ln x$ →

Thm1 if $x>0$

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

In general, $$\boxed{\frac{d}{dx}\ln u(x) = \frac{u'(x)}{u(x)}.}$$

Thm2 Properties of the Natural Logarithm

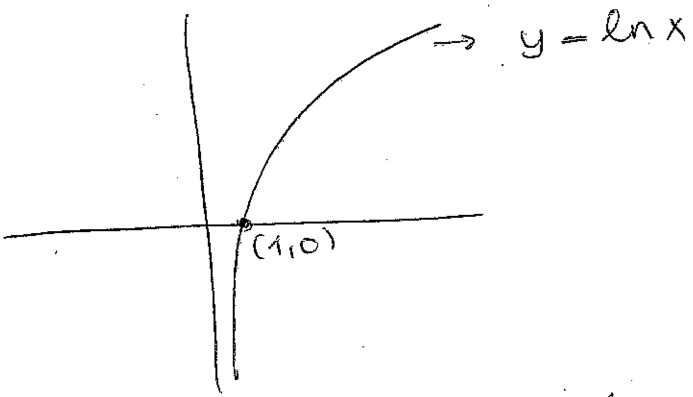
1-) $\ln(xy) = \ln x + \ln y$

2-) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

3-) $\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x = 0 - \ln x = -\ln x.$

4-) $\ln(x^r) = r\ln x.$

→ The Graph of $\ln x : (0,\infty) \to (-\infty,\infty)$



$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$

$\ln 0 = -\infty$

$\ln \infty = \infty.$

$\ln e = 1$

Ex1 Show that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for any $x \neq 0$

Hence find $\int \frac{1}{x}\,dx$.

Soln: if $x > 0$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$$

if $x < 0$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\ln|x| = \frac{1}{x} \text{ for any } x \neq 0.$$

$$\Rightarrow \boxed{\int \frac{1}{x}\,dx = \ln|x| + c.}$$

Ex2 Find the derivative of

38. a.) $y = \ln|\sin x|$

p.192

49. b.) $y = \ln\left(\sqrt{x^2+a^2} - x\right)$

c.) $y = \ln\left|\sqrt[3]{6x+7}\right|$

d.) $y = \frac{1}{\ln x} + \ln\frac{1}{x}$

e.) $y = \ln(\cos(2x))$

Soln:

a.) $$y' = \frac{1}{\sin x}\cos x = \frac{\cos x}{\sin x} = \cot x.$$

b.) $$y' = \frac{1}{\sqrt{x^2+a^2} - x}\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}} - 1\right) = \frac{1}{\cancel{\sqrt{x^2+a^2} - x}}\left(\frac{\cancel{x - \sqrt{x^2+a^2}}}{\sqrt{x^2+a^2}}\right)$$

$$= -\frac{1}{\sqrt{x^2+a^2}}$$

The Exponential Function :

$\ln x$ is a 1-to-1 function on $(0,\infty)$

$\Rightarrow$ it has an inverse, we call this inverse function $\exp x$.

$$\boxed{y = \exp x \iff x = \ln y \quad (y>0)}$$

$\ln 1 = 0 \quad \Rightarrow \quad \boxed{\exp 0 = 1}$

Domain of exp : $(-\infty,\infty)$ $\longrightarrow$ range of ln

Range of exp : $(0,\infty)$ $\longrightarrow$ domain of ln.

$$\boxed{\begin{array}{ll} \ln(\exp x) = x & \text{for all real } x. \\ \exp(\ln x) = x & \text{for } x>0. \end{array}}$$

Thm3 Properties of the Exponential Function

(1.) $(\exp x)^r = \exp(rx)$

(2.) $\exp(x+y) = (\exp x)(\exp y)$

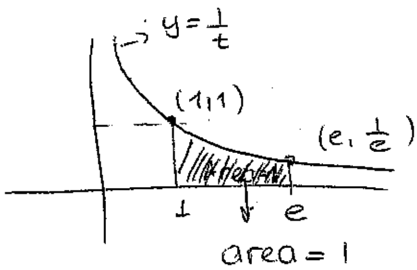
(3.) $\exp(-x) = \dfrac{1}{\exp(x)}$

(4.) $\exp(x-y) = \dfrac{\exp x}{\exp y}$

<u>Defn:</u>

Let $\boxed{e = \exp(1)}$

$\ln e = 1$ so



$e = 2.7182818 \ldots$

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

$\boxed{e^x = \exp x}$ for all real x.

then <u>thm 3</u> will be

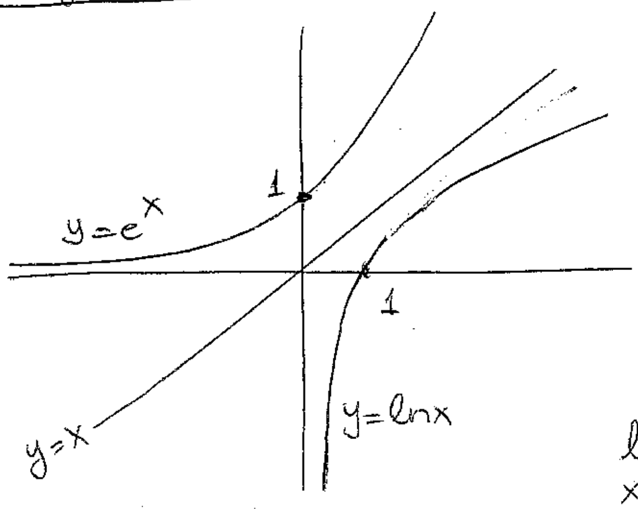
1-) $(e^x)^y = e^{xy}$

2.) $e^{x+y} = e^x . e^y$

3.) $e^{-x} = \frac{1}{e^x}$

4-) $e^{x-y} = \frac{e^x}{e^y}$

<u>Grph of $y = e^x$ and $\ln x$</u>



x-axis is horizontal asymptote for $y = e^x$ as $x \to -\infty$

$\lim_{x \to -\infty} e^x = 0$ $\lim_{x \to \infty} e^x = \infty$

Definition:

$$\boxed{\ln x = \log_e x}$$

Derivative of $y = e^x$.

$$y = e^x \Rightarrow x = \ln y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\ln y)$$

$$1 = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = y = e^x .$$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

in general $\boxed{\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)}$

$$\boxed{\int e^x \, dx = e^x + c}$$

Ex Find the derivatives of

(a-) $y = e^{x^2 - 3x}$.

$$y' = e^{x^2-3x} \cdot (2x-3) = (2x-3)\, e^{x^2-3x}$$

(b-) $y = x^2 e^{x/2}$

$$y' = 2x e^{x/2} + x^2 \cdot e^{\frac{x}{2}} \cdot \frac{1}{2} = 2x e^{x/2} + \frac{1}{2} x^2 e^{x/2} .$$

c.) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$y' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{\cancel{(e^x)^2} + 2e^x e^{-x} + \cancel{(e^{-x})^2} - \cancel{(e^x)^2} + 2e^x e^{-x} + \cancel{(e^{-x})^2}}{(e^x + e^{-x})^2}$$

$$y' = \frac{4}{(e^x + e^{-x})^2}$$

<u>Ex:</u> Let $f(t) = e^{at}$ Find

a.) $f^{(n)}(t)$.

$f'(t) = a e^{at}$

$f''(t) = a^2 e^{at}$

$\vdots$

$f^{(n)}(t) = a^n e^{at}$

b.) $\int f(t)\, dt$.

$\int e^{at}\, dt = \frac{1}{a} e^{at} + C$.

$\frac{d}{dt}\left(\frac{1}{a} e^{at}\right) = e^{at}$

d) $y = \sqrt{e^{2x} + 3x}$

e) $y = (e^{3x} - e^{-3x})^4$

f) $y = e^{\sqrt{x}} + \sqrt{e^x}$

General Exponentials and Logarithms:

Defn: The general Exponential a^x

$$\boxed{a^x = e^{x \ln a}} \qquad a > 0,\ x \text{ real}$$

Ex: Evaluate 2^π using the natural logarithm (ln and exponential (exp or e^x) keys on calculator.

$$2^\pi = e^{\pi \ln 2} = 8.82497\ldots$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} . \ln a = a^x \ln a.$$

$$\boxed{\frac{d}{dx} a^x = a^x \ln a} \qquad \boxed{\frac{d}{dx} a^{u(x)} = a^{u(x)} . u'(x) \ln a}$$

$\Rightarrow$ Ex.

Ex 2 Show that the graph of $f(x) = x^\pi - \pi^x$ has a negative slope at $x = \pi$.

Soln: $f'(x) = \pi x^{\pi - 1} - \pi^x \ln \pi$

$f'(\pi) = \pi \, \pi^{\pi - 1} - \pi^\pi \ln \pi = \pi^\pi (1 - \ln \pi)$

$e < 3 < \pi \quad \Rightarrow \quad \ln e < 3 < \ln \pi$

$1 < 3 < \ln \pi$

$\Rightarrow \quad 1 < \ln \pi \quad \Rightarrow \quad 1 - \ln \pi < 0.$

$\pi^{\pi} = e^{\pi \ln \pi} > 0$ (with $\pi \ln \pi > 1$)

so $f'(\pi) < 0 \Rightarrow f(x) = x^{\pi} - \pi^{x}$ has negative slope at $x = \pi$.

Ex: Find the critical point of $y = x^{x}$

Soln:

$$y = x^{x} = e^{x \ln x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x \ln x}\right) =$$

$$\frac{dy}{dx} = e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = e^{x \ln x}\left(\ln x + 1\right)$$

$$y' = \frac{dy}{dx} = x^{x}\left(\ln x + 1\right)$$

?? x^{x} defined for $x > 0$. and never zero.

so $y'(x) = 0$ when $\ln x + 1 = 0 \Rightarrow \ln x = -1$.

$$e^{\ln x} = e^{-1}$$

$$\Rightarrow \boxed{x = \frac{1}{e}}$$

Ex: $y = 2^{(x^2 - 3x + 8)}$

$$y' = 2^{(x^2-3x+8)} \cdot (2x-3) \cdot \ln 2 .$$

$\rightarrow \star$

Ex: $y = 8^{(\sqrt{x} + x.)}$

Ex: $y = 2^{\sec 3x}$

Soln: $y' = 2^{\sec 3x} \ln|2| . \sec 3x \tan 3x . 3$

Ex: $y = (10^{x} + 10^{-x})^{4}$

Remark: (1) $\frac{d}{dx} a^x = a^x \ln a$ < 0 for all x if $0 < a < 1$

> 0 " if $a > 1$

(2) $\Rightarrow$ a^x is 1-to-1 function and has inverse function.

$\log_a x$, where $a > 0$, $a \neq 1$

(3) if $y = \log_a x \Rightarrow a^y = x$

↓ take implicit differentiation.

$$\frac{d}{dx}(a^y) = \frac{d}{dx} x$$

$$a^y \ln a \cdot \frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{x \ln a}.$$

Ex: $y = \log (3x^2+2)^5$

Ex: $y = \log \sqrt{x^2+1}$

So $\boxed{\frac{d}{dx} \log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{1}{x}.}$

$$\boxed{\frac{d}{dx} \log_a u(x) = \frac{d}{dx}\left(\frac{\ln u(x)}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}.}$$

→ Ex 2

Logarithmic Differentiation:

. Suppose we want to differentiate a function of the form

$$y = (f(x))^{g(x)} \quad ; \quad f(x) > 0$$

Ist Method: (used in previous example).

1) Express function as. $y = e^{g(x) \ln f(x)}$

2.) Differentiate using the Product Rule.

II nd Method : (Logarithmic Differentiation)

1.) Take the natural logarithms of both sides of eqn

2.) Differentiate implicitly.

Ex:

$$y = x^x$$ $\quad$ Ex: $y = x^{2x}$

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = x^x(\ln x + 1).$$

Ex: Find $f'(t)$ if $f(t) = (\sin t)^{\ln t}$ $\quad 0 < t < \pi$

Soln:

$$y = (\sin t)^{\ln t}$$

$$\ln y = \ln(\sin t)^{\ln t} = \ln t \ln \sin t$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{t}\ln \sin t + \ln t \frac{1}{\sin t} \cdot \cos t.$$

$$\frac{dy}{dx} = y\left(\frac{\ln \sin t}{t} + \ln t \cot t\right)$$

$$\frac{dy}{dx} = (\sin t)^{\ln t}\left(\frac{\ln \sin t}{t} + \ln t \cot t\right)$$

Remark: Logarithmic differentiation is also useful for finding the derivatives of functions expressed as products and quotients of many factors.

Because taking logarithms reduces these products and quotients to sums and differences.

Ex: Differentiate

$$y = [(x+1)(x+2)(x+3)]/(x+4).$$

$$\ln|y| = \ln\left([(x+1)\cdot(x+2)(x+3)]/(x+4)\right)$$

$$\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$$

$$\frac{1}{y}\cdot y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

$$y' = \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right).$$

$$y' = \frac{(x+2)(x+3)}{x+4} + \frac{(x+1)(x+3)}{x+4} + \frac{(x+1)(x+2)}{x+4} - \frac{(x+1)(x+2)(}{(x+4)}.$$

Ex: Find $\left.\frac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$

$$\ln u = \ln[(x+1)(x^2+1)(x^3+1)]^{1/2}$$

$$\ln u = \frac{1}{2}\left[\ln (x+1)(x^2+1)(x^3+1)\right]$$

$$\ln u = \frac{1}{2}\left[\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\right]$$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{2}\left[\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1}\right]$$

$$x=1 \Rightarrow u = \sqrt{8} = 2\sqrt{2}$$

$$\left.\frac{du}{dx}\right|_{x=1} = \cancel{2}\sqrt{2}\cdot\frac{1}{\cancel{2}}\left(\frac{1}{2}+1+\frac{3}{2}\right) = 3\sqrt{2} \,//$$

P192 58, 59, 44.

The Inverse Trigonometric Functions :

44/ $y = \log_x(2x+3) \qquad y' = ?$

$$y = \log_x(2x+3) \iff x^y = 2x+3$$

$$\frac{d}{dx}(x^y) = \frac{d}{dx}(2x+3)$$

$$x^y\cdot y'\ln x + \frac{y}{x} = 2.$$

$$z = x^y$$

$$z = e^{y\ln x}$$

$$\frac{dz}{dx} = \frac{d}{dx}\left(e^{y\ln x}\right)$$

$$= e^{y\ln x}\left(\frac{dy}{dx}\ln x + \frac{1}{x}y\right)$$

$$= x^y\left(y'\ln x + \frac{1}{x}y\right)$$

$$y' = \left(2-\frac{y}{x}\right)\frac{1}{x^y\ln x}$$

$$= \left(2 - \frac{1}{x}\log_x(2x+3)\right)\frac{1}{(2x+3)\ln x}$$

$$=$$

Sec 3.5 The Inverse Trigonometric Function

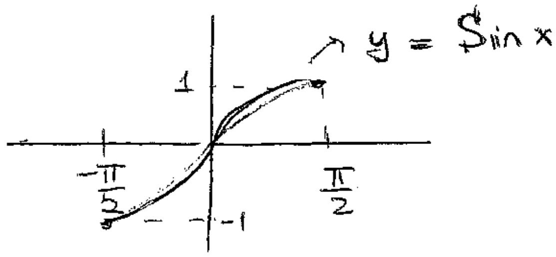
Trigonometric functions $\rightarrow$ are periodic $\rightarrow$ not one-to-one.

we can restrict their domains $\rightarrow$ restricted functions are 1-to-1 $\rightarrow$ invertible.

The Inverse Sine (Arcsine) Function

Defn: The restricted function $\text{Sin}\,x$

$$\text{Sin}\,x = \sin x \quad \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$



Domain $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range $= [-1, 1]$

$y = \text{Sin}\,x$

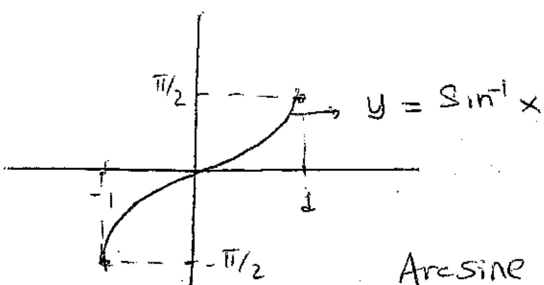
$y' = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\Rightarrow$ $\text{Sin}\,x$ is increasing $\Rightarrow$ 1-to-1

$\Rightarrow$ it has inverse,

Defn: The Inverse sine function $\sin^{-1} x$ or $\arcsin x$

$$y = \sin^{-1} x \iff x = \text{Sin}\,y$$

$$\boxed{y = \sin^{-1} x \iff x = \sin y \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}}$$



$\text{dom}(\text{Sin}^{-1} x) = [-1, 1]$

$\text{range}(\text{Sin}^{-1} x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Arcsine function.

Cancellation Identities :

$$\boxed{\begin{aligned} \sin^{-1}(\sin x) &= \arcsin(\sin x) = x \\ \sin(\sin^{-1} x) &= \sin(\arcsin x) = x \end{aligned}}$$

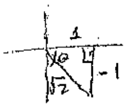
$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$-1 \le x \le 1$

Ex:

✓ a.) $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$

b.) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{-\pi}{4}$ $\quad -\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$.



✓ c.) $\sin^{-1} 2$ not defined.

✓ d.) $\sin(\sin^{-1} 0.7) = 0.7$ $\quad -1 < 0.7 < 1$

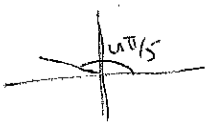
e.) $\sin^{-1}(\sin 0.3) = 0.3$ $\quad \cancel{-1 < 0.3 < 1}$

↓ Not possible

✓ f.) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right) = \frac{4\pi}{5}$ but $-\frac{\pi}{2} \le \frac{4\pi}{5} \le \frac{\pi}{2}$

so we can't use cancellation ident

$$\sin^{-1}\left(\sin\frac{4\pi}{5}\right) = \sin^{-1}\left(\sin\frac{\pi}{5}\right) = \frac{\pi}{5} \qquad -\frac{\pi}{2} < \frac{\pi}{5} < \frac{\pi}{2}$$



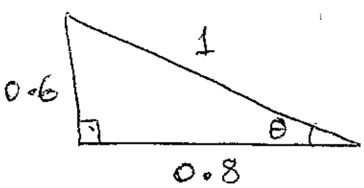
$$\sin\frac{4\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) = \sin\frac{\pi}{5}$$

Ex: Simplify

a.) $\cos(\sin^{-1} 0.6)$

Let $\theta = \sin^{-1} 0.6 \Rightarrow \sin\theta = 0.6$.

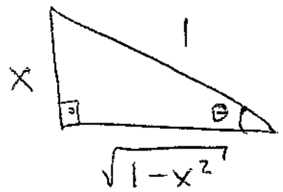
$\cos(\sin^{-1} 0.6) = 0.8$



(b.) $\tan(\sin^{-1} x)$

suppose $0 \le x \le 1$

Let $\theta = \sin^{-1} x \Rightarrow \sin\theta = x$.



$$\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

odd funct   odd funct.

symmetric about y-axis.

we have same result when $-1 < x < 0$

— o —

<u>Derivative of inverse sine function</u>

if $y = \sin^{-1} x \Rightarrow \sin y = x \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

by implicit diff. $(\cos y)\frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y > 0$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}}$$

In general

$$\boxed{\frac{d}{dx}\sin^{-1} u(x) = \frac{d}{dx}\arcsin u(x) = \frac{u'(x)}{\sqrt{1-(u(x))^2}}}$$

<u>Note that:</u> $\sin^{-1} x$ is differentiable only on the open interval $(-1,1$

Ex: Find derivative of $\sin^{-1}\left(\frac{x}{a}\right)$ and evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$ where $a>0$.

Soln:

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{\frac{1}{a}}{\sqrt{1-\frac{x^2}{a^2}}} = \frac{\frac{1}{a}}{\sqrt{\frac{a^2-x^2}{a^2}}} = \frac{\frac{1}{\cancel{a}}}{\frac{1}{\cancel{a}}\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}}$$

$a>0$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C.}$$

$a>0$.

Ex: $\sin^{-1}\left(\frac{1}{3}y\right)$

$$\frac{d}{dy}\sin^{-1}\frac{1}{3}y = \frac{\frac{1}{3}}{\sqrt{1-\frac{1}{9}y^2}} = \frac{1}{3}\frac{1}{\sqrt{\frac{9-y^2}{9}}}$$

Ex: $\sin^{-1}(\sqrt{x})$

$$= \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}$$

Ex: Find solution y of the following I.V.P.

$$y' = \frac{4}{\sqrt{2-x^2}} \qquad (-\sqrt{2} < x < \sqrt{2})$$

$$y(1) = 2\pi$$

Soln:

$$y = 4\int \frac{1}{\sqrt{2-x^2}}\,dx = 4\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$y(1) = 4\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + C = 2\pi$$

$$\cancel{4}\cdot\frac{\pi}{\cancel{4}} + C = 2\pi \quad \Rightarrow \quad C = \pi$$

$$y = 4\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + \pi$$

NO

Ex: Let $f(x) = \sin^{-1}(\sin x)$ for all real x.

a.) Calculate and simplify $f'(x)$.

b.) Where is f differentiable? Where is f continuous?

c.) Use your results from (a) and (b) to sketch the graph of f

Soln:

a.) $$f'(x) = \frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|} = \begin{cases} 1 & \text{if } \cos x \\ -1 & \text{if } \cos x < \end{cases}$$

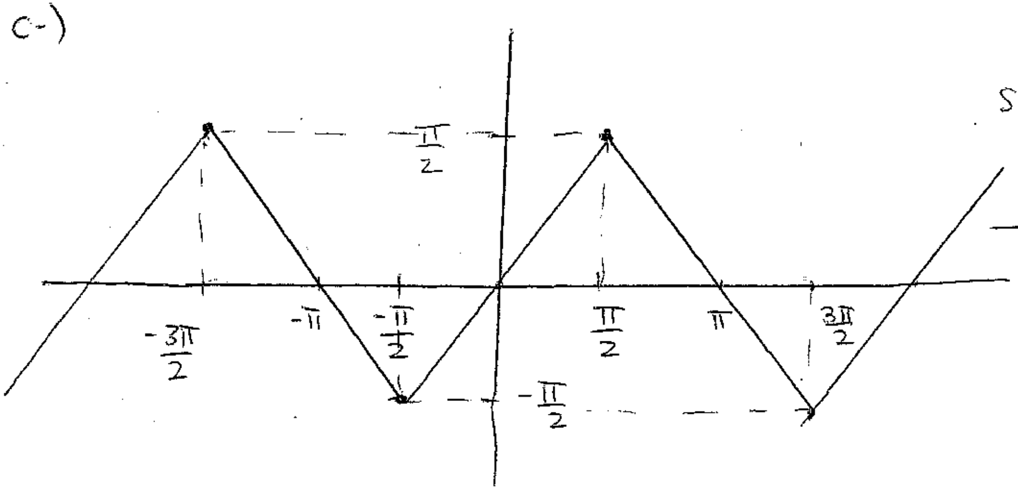
b.) f is differentiable at all points where $\cos x \neq 0$

ie/ " everywhere except at $x = k\frac{\pi}{2}$ $k = \mp 1, \mp 3, \mp 5, \mp 7 \ldots$

$\mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \mp\frac{5\pi}{2} \ldots$

sin is continuous everywhere has values in $[-1,1]$ and $\sin^{-1}$ is continuous on $[-1,1]$ $\Rightarrow$ f is cont on whole real line.

c.)



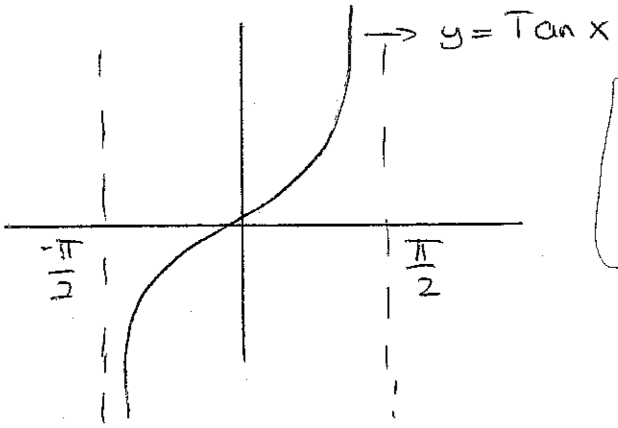
$\sin^{-1}\left(\sin\frac{3\pi}{2}\right) = \sin^{-1}-1 = -\frac{\pi}{2}$

$\rightarrow y = \sin^{-1}(\sin x)$.

The Inverse Tangent (Arctangent) Function

Defn: The restricted function Tan x

$$\text{Tan } x = \tan x \quad \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



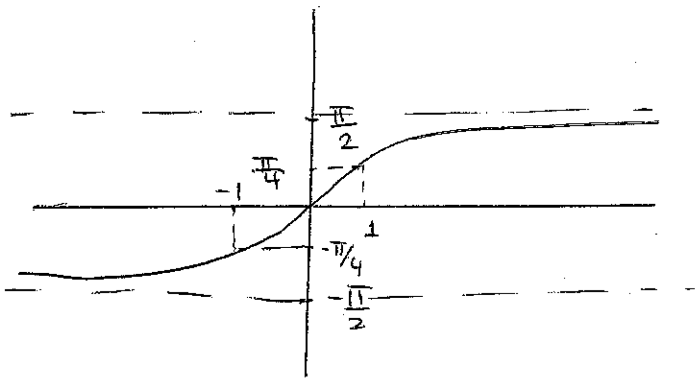
$$\boxed{\begin{array}{l} \text{dom}(\text{Tan } x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \text{range}(\text{Tan } x) = \mathbb{R} \end{array}}$$

1-1 function.

Defn: The inverse tangent function $\tan^{-1} x$ or $\arctan x$

$$y = \tan^{-1} x \iff x = \text{Tan } y$$

$$\boxed{y = \tan^{-1} x \iff x = \tan y \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}}$$



$$\boxed{\begin{array}{l} \text{dom}(\tan^{-1} x) = \mathbb{R} \\ \text{range}(\tan^{-1} x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}}$$

Cancellation Identities:

$$\boxed{\begin{array}{l} \tan^{-1}(\tan x) = \arctan(\tan x) = x \\ \tan(\tan^{-1} x) = \tan(\arctan x) = x \end{array}}$$

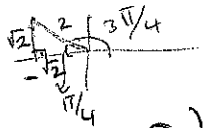
$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\underbrace{-\infty < x < \infty}_{x \in \mathbb{R}}$

Ex: Evaluate

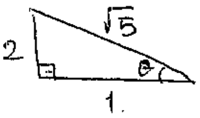
a.) $\tan(\tan^{-1} 3) = 3$ $\qquad -\infty < 3 < \infty$

b.) $\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$ $\qquad -\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$.



c.) $\cos(\tan^{-1} 2) =$

$\theta = \tan^{-1} 2 \Rightarrow \tan\theta = \frac{2}{1}$.



$\cos(\tan^{-1} 2) = \frac{1}{\sqrt{5}}$

Derivative of Inverse tangent function:

if $y = \tan^{-1} x \Rightarrow \tan y = x$

$(\sec^2 y) \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}}$$

In general, $\boxed{\frac{d}{dx} \tan^{-1} u(x) = \frac{u'(x)}{1+(u(x))^2}}$

Ex: Find $\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right)$ and evaluate $\int \frac{1}{x^2+a^2}\,dx$

Soln:

$$\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{\frac{1}{a}}{1+\frac{x^2}{a^2}} = \frac{\frac{1}{a}}{\frac{a^2+x^2}{a^2}} = \frac{a}{a^2+x^2}$$

$$\boxed{\int \frac{a}{x^2+a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + C.}$$

Ex: $\frac{d}{dx} \tan^{-1}(3x-5) = ?$

$= \frac{3}{1+(3x-5)^2}$

$= \frac{3}{9x^2-30x+26}$

$$\Rightarrow a \int \frac{1}{x^2+a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow \boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.}$$

Ex: $\frac{d}{dx} \tan^{-1}(x^2)$

$= \frac{2x}{1+x^4}$

Ex: Prove that $\tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}x - \frac{\pi}{4}$ $\quad x > -1$

Soln:

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}x = -\frac{\pi}{4}$$

Ex: $\frac{d}{dx}(\ln \tan^{-1}x^2)$

$\frac{1}{\tan^{-1}x^2} \cdot \frac{2x}{1+x^4}$

to show two sides are equal.

Let $f(x) = \tan^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}x$

$$f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} \cdot \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} - \frac{1}{1+x^2}$$

$$= \frac{2}{(x+1)^2+(x-1)^2} - \frac{1}{1+x^2} = \frac{\cancel{2}}{\cancel{2}(x^2+1)} - \frac{1}{1+x^2} = 0.$$

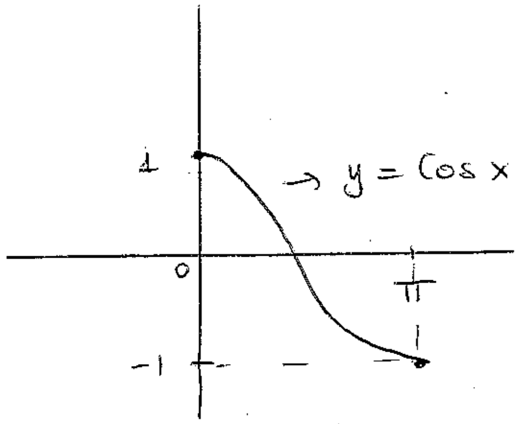
$f'(x) = 0 \Rightarrow f(x) = C$ constant.

we can find c by finding f(0)

$$f(0) = \tan^{-1}(-1) - \tan^{-1}(0) = -\frac{\pi}{4} = C. \qquad x > -1$$

So given identity is true.

Other Inverse Trigonometric Functions:



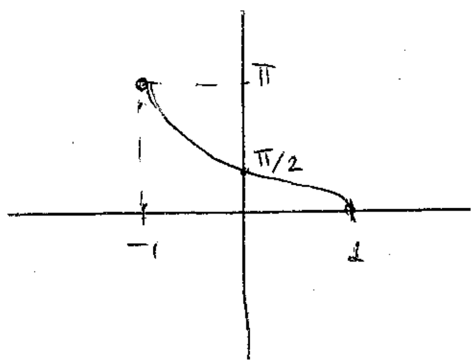
$\boxed{\text{dom}(\cos x) = [0, \pi] \quad \text{range}(\cos x) = [-1, 1]}$

on $[0, \pi]$ $\cos x$ is 1-to-

Defn: Inverse Cosine function, $\cos^{-1} x$ or $\text{ArcCos}\, x$

$$\boxed{y = \cos^{-1} x \iff x = \cos y \text{ and } 0 \leq y \leq \pi}$$

$0 \leq \cos^{-1} x \leq \pi$



$\cos y = \sin\left(\frac{\pi}{2} - y\right)$ $\qquad \frac{\pi}{2} - y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ when $0 \le y \le \pi$

So,

$$y = \cos^{-1}x \iff x = \sin\left(\frac{\pi}{2} - y\right) \Rightarrow \sin^{-1}x = \frac{\pi}{2} - y = \frac{\pi}{2} - \text{Co}$$

$$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x.$$

$$\boxed{\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \quad \text{for } -1 \le x \le 1}$$

$$\boxed{\begin{array}{ll} \cos^{-1}(\cos x) = \arccos(\cos x) = x & \text{for } 0 \le x \le \pi \\ \cos(\cos^{-1}x) = \cos(\arccos x) = x & \text{for } -1 \le x \le 1 \end{array}}$$

<u>derivative of $\cos^{-1}x$:</u>

$$\boxed{\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}}$$

Ex: $\frac{d}{dx}\left(\cos^{-1}\cos e^x\right) = \frac{-(-\sin e^x)\cdot e^x}{\sqrt{1-\cos^2 e^x}}$

In general

$$\boxed{\frac{d}{dx}\cos^{-1}u(x) = \frac{-u'(x)}{\sqrt{1-(u(x))^2}}}$$

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Q21

<u>Ex:</u> Differentiate $y = \cos^{-1}\left(\frac{x-b}{a}\right)$

<u>Soln:</u>

$$y' = \frac{-\frac{1}{a}}{\sqrt{1-\left(\frac{x-b}{a}\right)^2}} = \frac{-\frac{1}{a}}{\sqrt{\frac{a^2-(x-b)^2}{a^2}}} = \frac{-\frac{1}{a}}{|a|\sqrt{a^2-(x-b)^2}}$$

Q25 (No) **Ex:** Differentiate $f(x) = (1+x^2)\tan^{-1}x$

Soln: $f'(x) = 2x\,(\tan^{-1}x) + (1+\cancel{x^2})\dfrac{1}{1+\cancel{x^2}}$

$f'(x) = 2x\,(\tan^{-1}x) + 1$

Q39 Exercise

Ex: Find the slope of the curve $\tan^{-1}\left(\frac{2x}{y}\right) = \frac{\pi x}{y^2}$ at the point $(1,2)$.

Soln:

$$\frac{\frac{d}{dx}\left(\frac{2x}{y}\right)}{1+\left(\frac{2x}{y}\right)^2} = \frac{d}{dx}\left(\pi x y^{-2}\right)$$

$$\frac{1}{1+\frac{4x^2}{y^2}}\left[\frac{2y - \frac{dy}{dx}2x}{y^2}\right] = \pi y^{-2} - 2\pi x y^{-3}\frac{dy}{dx}$$

at $(1,2)$

$$\frac{1}{1+\frac{4}{4}}\left[\frac{4 - \frac{dy}{dx}\Big|_{x=1}\cdot 2}{4}\right] = \frac{\pi}{4} - \frac{2\pi}{8}\frac{dy}{dx}\Big|_{x=1}$$

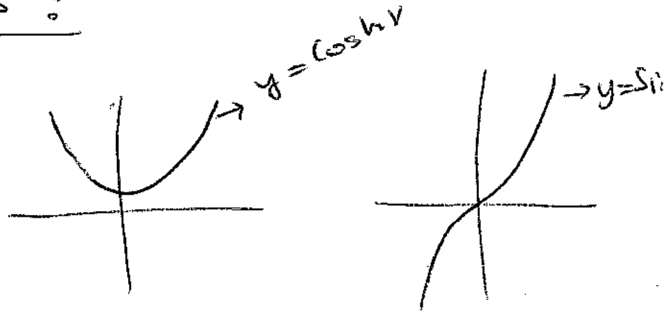
$$\frac{1}{2}\left[1 - \frac{1}{2}\frac{dy}{dx}\Big|_{x=1}\right] = \frac{\pi}{4} - \frac{\pi}{4}\frac{dy}{dx}\Big|_{x=1}$$

$$\frac{1}{2} - \frac{1}{4}\frac{dy}{dx}\Big|_{x=1} + \frac{\pi}{4}\frac{dy}{dx}\Big|_{x=1} = \frac{\pi}{4}$$

$$\frac{\pi-1}{4}\frac{dy}{dx}\Big|_{x=1} = \frac{\pi}{4} - \frac{2}{4}$$

$$\frac{dy}{dx}\Big|_{x=1} = \frac{\pi-2}{\pi-1}$$

Sec 3.6 Hyperbolic Functions :



Defn:

For any real x , the

Hyperbolic Cosine, Cosh x

$$\boxed{\cosh x = \frac{e^{x}+e^{-x}}{2}} \rightarrow \text{Even funct.}$$

Hyperbolic Sine, Sinh x

$$\boxed{\sinh x = \frac{e^{x}-e^{-x}}{2}} \rightarrow \text{Odd funct.}$$

Cosh and Sinh are called <u>hyperbolic functions</u> because point (Cosht, Sinht) lies on the rect. hyperbola with eqn $x^2-y^2=1$.

$$\boxed{\cosh^2 t - \sinh^2 t = 1}$$ for any real t.

* $$\boxed{\begin{aligned} \cosh 0 &= 1 \\ \sinh 0 &= 0 \end{aligned}}$$

* $$\boxed{\begin{aligned} \cosh(-x) &= \cosh x \rightarrow \text{even funct.} \\ \sinh(-x) &= -\sinh x \rightarrow \text{odd funct.} \end{aligned}}$$

Ex: Show that

a.) $\boxed{\frac{d}{dx} \cosh x = \sinh x}$

b.) $\boxed{\frac{d}{dx} \sinh x = \cosh x}$

Soln:

a) $\frac{d}{dx} \cosh x = \frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{e^{x}-e^{-x}}{2} = \sinh x$

b) $\frac{d}{dx} \sinh x = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right) = \frac{e^{x}+e^{-x}}{2} = \cosh x$

Properties:

1.) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

2.) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

3.) $\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + \sinh^2 x + \sinh^2 x = 1+2\sinh^2 x$

or/ $\cosh(2x) = 2\cosh^2 x - 1$

4.) $\sinh(2x) = 2\sinh x \cosh x$

Defn:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}$$

$\sqrt{x^2+1} > x$ and e^y cannot be negative.

$\Rightarrow \quad e^y = x + \sqrt{x^2+1}$

$y = \ln(x + \sqrt{x^2+1})$

$$\boxed{\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)}$$

Let $\quad y = \tanh^{-1} x \quad \Rightarrow \quad x = \tanh^{-1} \cancel{x}\, y = \dfrac{(e^y - e^{-y})\, e^y}{(e^y + e^{-y})\, e^y}$

$$x = \frac{(e^y)^2 - 1}{(e^y)^2 + 1} \qquad -1 < x < 1$$

$x e^{2y} + x = e^{2y} - 1$

$e^{2y} = \dfrac{x+1}{1-x}$

$\ln e^{2y} = \ln\left(\dfrac{x+1}{1-x}\right) \Rightarrow 2y = \ln\left(\dfrac{x+1}{1-x}\right)$

$y = \frac{1}{2} \ln\left(\dfrac{x+1}{1-x}\right)$

$$\boxed{\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)} \qquad -1 < x < 1$$

<u>Inverse of $\cosh x$</u> :

Cosh is not 1-to-1 so its domain must be restricted.

$y = \cosh^{-1} x \quad \Leftrightarrow \quad x = \cosh y \quad (y \geq 0)$

$\cosh x = \cosh$

$x \geq 0$

In the same way

$$\boxed{\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geq 1}$$

$$\frac{d}{dx}\tanh x = \operatorname{Sech}^2 x \qquad \frac{d}{dx}\operatorname{Sech} x = -\operatorname{Sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{Coth} x = -\operatorname{Csch}^2 x \qquad \frac{d}{dx}\operatorname{Csch} x = -\operatorname{Csch} x \operatorname{Coth} x$$

Ex: $\frac{d}{dx}\sinh(x^2+1) = \cosh(x^2+1)\cdot 2x$ $\qquad$ $\frac{d}{dx}\cosh^3 x = 3\cosh^2 x \sinh x$

Ex: show that $\frac{d}{dx}\tanh x = \operatorname{Sech}^2 x$

Soln: $$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x}$$

Ex: $\frac{d}{dx}(\arctan \tanh x)$

$$= \frac{1}{\cosh^2 x} = \operatorname{Sech}^2 x.$$

$\frac{1}{1+\tanh^2 x}\cdot \operatorname{sech}^2 x.$

Ex: $\frac{d}{dx}\sqrt{\operatorname{sech} 5x} = \frac{1}{2}(\operatorname{sech} 5x)^{-1/2}\cdot(-\operatorname{sech} 5x \tanh 5x)\cdot 5$

Inverse Hyperbolic Functions: (??)

$$y = \sinh^{-1} x \iff x = \sinh y$$

$$y = \tanh^{-1} x \iff x = \tanh y$$

Ex: Express $\sinh^{-1} x$ and $\tanh^{-1} x$ in terms of logarithms.

Soln: Let $y = \sinh^{-1} x \Rightarrow x = \sinh y = \left(\frac{e^y - e^{-y}}{2}\right)\frac{e^y}{e^y}$

$$x = \frac{(e^y)^2 - 1}{2e^y}$$

$$(e^y)^2 - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \mp \sqrt{4x^2+4}}{2} = x \mp \sqrt{x^2+1}$$

p.215 Exercise

① **Ex:** Verify that $\frac{d}{dx}\text{Sech}\,x = -\text{Sech}\,x\tanh x$

Soln:

$$\frac{d}{dx}\text{Sech}\,x = \frac{d}{dx}\left(\frac{1}{\text{Cosh}x}\right) = \frac{d}{dx}\left(\frac{2}{e^{x}+e^{-x}}\right)$$

$$= \frac{0-2(e^{x}-e^{-x})}{(e^{x}+e^{-x})^2} = \frac{-2}{e^{x}+e^{-x}}\cdot\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = -\text{Sech}\,x\,\text{ta}$$

5. (No)

Ex: Find

a.) $\frac{d}{dx}\text{Sinh}^{-1}x$ and $\frac{d}{dx}\tanh^{-1}x$

b.) Express each of the indefinite integrals in terms of inverse hyperbolic functions.

$$\int\frac{dx}{\sqrt{x^2+1}}, \quad \int\frac{dx}{1-x^2}.$$

Soln: $\text{Sinh}^{-1}x = \ln(x+\sqrt{x^2+1})$

$$\frac{d}{dx}\text{Sinh}^{-1}x = \frac{d}{dx}\ln(x+\sqrt{x^2+1}) = \frac{1+\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot\cancel{2}x}{x+\sqrt{x^2+1}}$$

$$= \frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\cdot\frac{1}{\cancel{x+\sqrt{x^2+1}}} = \frac{1}{\sqrt{x^2+1}}$$

So $\boxed{\int\frac{dx}{\sqrt{x^2+1}} = \text{Sinh}^{-1}x + C.}$

$$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\left(\frac{1+x}{1-x}\right)^{1/2}.$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{\cancel{\frac{1}{2}}\left(\frac{1+x}{1-x}\right)^{-1/2}\cdot\frac{(1-\cancel{x})+(1+\cancel{x})}{(1-x)^2}}{\left(\frac{1+x}{1-x}\right)^{1/2}} = \frac{1}{(1-\cancel{x})^2_{(1-x)}}\cdot\frac{1-\cancel{x}}{(1+x)}$$

$$= \frac{1}{1-x^2}$$

$$\boxed{\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}}$$

$$\boxed{\int\frac{dx}{1-x^2} = \tanh^{-1}x + c}$$

Q7

✓ Ex: Simplify:

$$\frac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x}$$

Soln:

$$= \frac{\frac{e^{\ln x}+e^{-\ln x}}{2} + \frac{e^{\ln x}-e^{-\ln x}}{2}}{\frac{e^{\ln x}+e^{-\ln x}}{2} - \frac{e^{\ln x}-e^{-\ln x}}{2}}$$

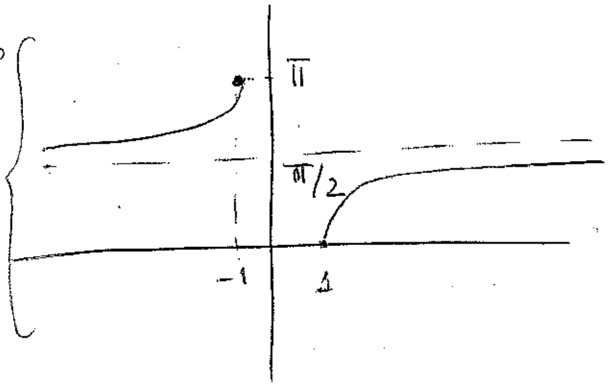
$$= \frac{\frac{x+\frac{1}{x}}{2} + \frac{x-\frac{1}{x}}{2}}{\frac{x+\frac{1}{x}}{2} - \frac{x-\frac{1}{x}}{2}} = \frac{\frac{2x^2}{2x}}{\frac{2}{2x}} = x^2$$

//

Defn: The inverse Secant function $\sec^{-1}x$ ($\operatorname{arcsec} x$)

$$\boxed{\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)} \quad \text{for } |x| \geq 1$$

No



$\pi/2 \cdot \cos^{-1}(-1) = \pi$ Ca

$y = \sec^{-1}x$

$\operatorname{Dom}(\sec^{-1}x) = (-\infty, -1] \cup [1, \infty)$

$\operatorname{Range}(\sec^{-1}x) = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

$$\sec(\sec^{-1}x) = \sec\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cos\left(\cos^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x \qquad |x| \geq$$

NO

$$\sec^{-1}(\sec x) = \cos^{-1}\left(\frac{1}{\sec x}\right) = \cos^{-1}(\cos x) = x \quad , x \in [0, \pi]$$

$$x \neq \frac{\pi}{2}$$

Derivative of $\sec^{-1}x$:

$$\frac{d}{dx}\sec^{-1}x = \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}}\cdot\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\sqrt{\frac{x^2}{x^2-1}} = \frac{1}{x^2}\frac{|x|}{\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\boxed{\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}} \qquad \boxed{\frac{d}{dx}\sec^{-1}u(x) = \frac{u'(x)}{u(x)\sqrt{(u(x))^2-1}}}$$

$$\boxed{\int \frac{1}{x\sqrt{x^2-1}}\,dx = \begin{cases} \sec^{-1}x + C & \text{when } x \geq 1 \\ -\sec^{-1}x + C & \text{when } x \leq -1 \end{cases}}$$

Defn:

$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right) \qquad |x| \geq 1$

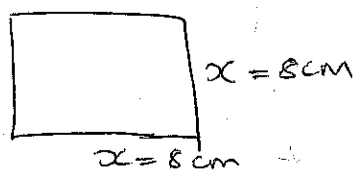
$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right) \qquad x \neq 0$.

Chapter 4 Some Applications of Derivatives

Sec 4.1 Related Rates

Ex 1 Find the rate of change of the area of a square whose side is 8 cm long, if the side length is increasing at 2 cm/min.

Soln:



$$A = x^2 \qquad \frac{dA}{dt} = ?$$

$$\frac{dx}{dt} = 2 \text{ cm/min}.$$

Take implicit diff

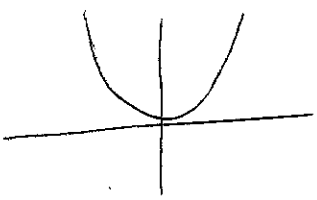
$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dA}{dt} = 2(8)(2) = 32 \text{ cm}^2/\text{min}.$$

The area of the square will increase at the rate of 32 cm^2 per minute.

Ex 2 A point moves on the curve $y = x^2$. How fast is y changing when $x = -2$ and x is decreasin at a rate 3?

Soln:



$$y = x^2 \qquad \frac{dy}{dt} = ?$$

$$x = -2$$

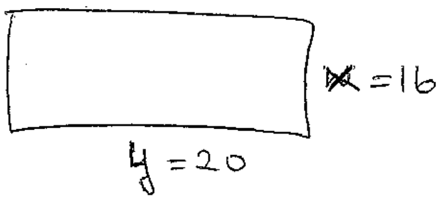
$$\frac{dx}{dt} = -3$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} = 2(-2)(-3) = 12$$

The value of y will increase at the rate of 12.

Ex3 The area of a rectangle is increasing at a rate of $5\,m^2/s$ while the length is increasing at a rate of $10\,m/s$. If the length is $20m$ and the width is $16m$, how fast is the width changing?

Soln:



$$\frac{dA}{dt} = 5$$

$$\frac{dy}{dt} = 10$$

$$\frac{dx}{dt} = ?$$

$$A = xy$$

$$\frac{dA}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$

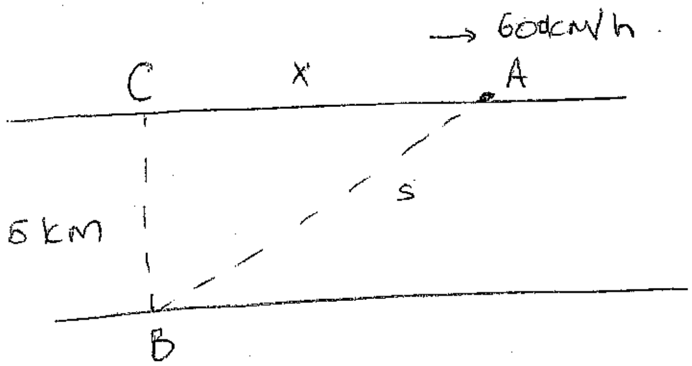
$$5 = \left.\frac{dA}{dt}\right|_{\substack{x=16\\y=20}} = 20\frac{dx}{dt} + 16(10) = 20\frac{dx}{dt} + 160$$

$$\frac{dx}{dt} = \frac{-155}{20} = -7.75$$

The width is decreasing at a rate 7.75.

Ex: An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 minute after the aircraft passes 5 km directly above the beacon?

Soln:



$s^2 = x^2 + 25$

$\frac{dx}{dt} = 600\,km/h = 10\,km/min$

$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$

$x = 10\,km$ at $t = 1\,min$.

$s = \sqrt{25+10^2} = 5\sqrt{5}\,km$: when t=

$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$

$\frac{ds}{dt} = \frac{10}{5\sqrt{5}} \cdot 600 \cong 536.7\,km/h$

Ex: Airi is being pumped into a spherical balloon. The volume of the baloon is increasing at a rate of $20\,cm^3/s$ when the radius is 30 cm. How fast is the radius increasing at that time?

Soln:

$\frac{dV}{dt} = 20 \text{ cm}^3/\text{s}$ when $r = 30$ cm.

$\frac{dr}{dt} = ?$

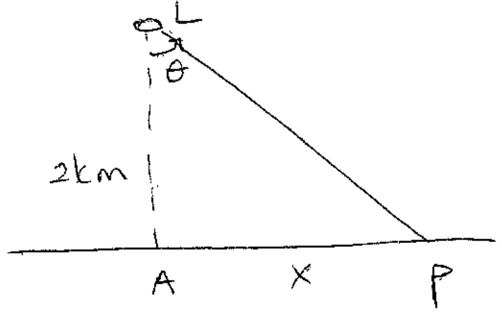
$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 20 = 4\pi \cdot 900 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{180\pi} \approx 0.0017$ cm/s.

Radius is increasing at $\frac{1}{180\pi} \approx 0.0017$ cm/s when it is 30 cm.

Ex: A lighthouse L is located on a small island 2 km from the nearest point A on a long, straight shoreline. If the lighthouse lamp rotates at 3 revoluti[ons] per minute, how fast is the illuminated spot P on the shoreline moving along the shoreline when it is 4 km from A?



$\tan\theta = \frac{x}{2}$

$\Rightarrow x = 2\tan\theta$

$\frac{dx}{dt} = 2\sec^2\theta \frac{d\theta}{dt}$

$$\frac{d\theta}{dt} = 3 \text{ rev/min} \times 2\pi \text{ radians/rev} = 6\pi \text{ radians/min.}$$

$x = 4$ $\qquad 4 = 2\tan\theta \Rightarrow \tan\theta = 2$

$$\sec^2\theta = 1 + \tan^2\theta = 5.$$

$$\frac{dx}{dt} = 2 \cdot 5 \cdot 6\pi = 60\pi \approx 188.5.$$

The spot of light is moving along the shoreline at a rate of about 188.5 km/min when it is 4 km from A

Sec 4.2 Extreme Values:

Maximum and Minimum Values:

Defn: Absolute extreme values

f has an absolute max val $f(x_0)$ at the point x_0 in its domain if $\boxed{f(x) \le f(x_0)}$ for $\forall x \in D_f$

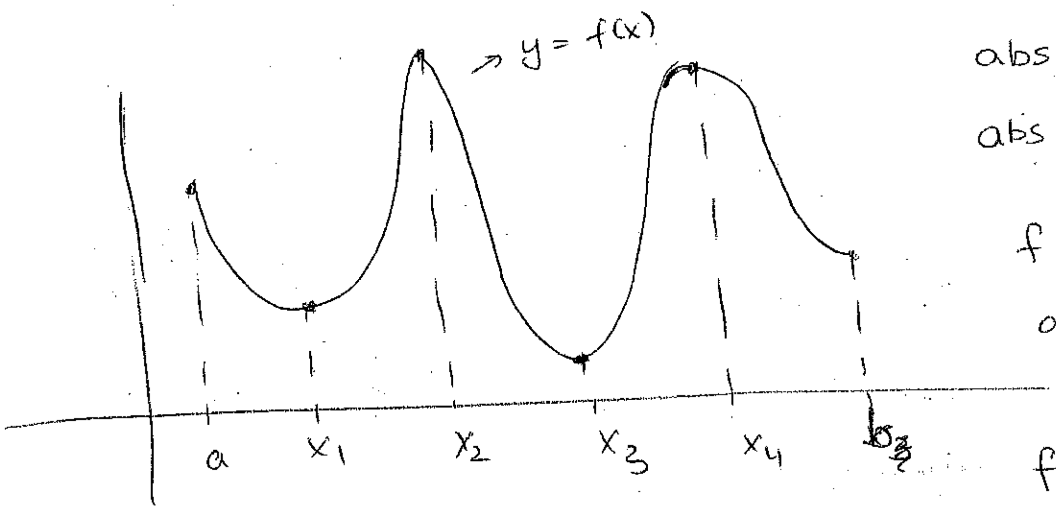
f has an absolute min val $f(x_1)$ at the point x_1 in its domain if $\boxed{f(x) \ge f(x_1)}$ for $\forall x \in D_f$.

There is at most one abs max or min.

Thm1 Existence of extreme values

If domain of f is closed, finite interval or union of finitely many such intervals and if f is continuous on that domain

$\Rightarrow$ f must have an abs max val and abs min val.



abs max of $f = f(x_2)$

abs min $= f(x_3)$

f has local max at a, x_2, x_4

f has local min at x_1, x_3, b

Defn: Local extreme values

f has local ~~extre~~ max val ~~at~~ $f(x_0)$ at $x_0 \in D_f$

if $\exists$ a $h > 0$ s.t $f(x) \le f(x_0)$ whenever $x \in D_f$
and $|x - x_0| < h$.

f has local min val $f(x_1)$ at $x_1 \in D_f$

if $\exists$ $h > 0$ s.t $f(x) \geq f(x_1)$ whenever $x \in D_f$
$|x - x_1| < h$

Critical Points, Singular Points and Endpoints

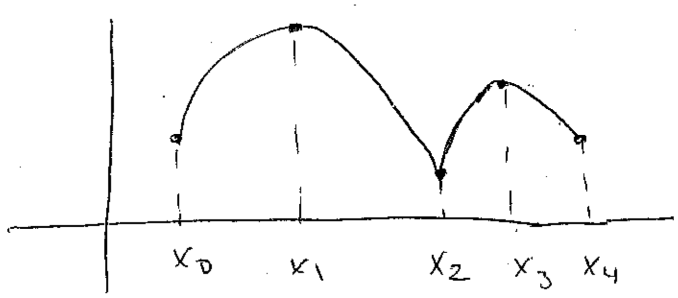
A function $f(x)$ has a local extreme values at x

($x \in \text{dom} f$) if

(1.) f has a critical point at x ($f'(x) = 0$)

(2.) f has singular point at x ($f'(x)$ is not defined

(3.) f has end-point of the domain of f.



x_0, x_4 are endpoints.

x_2 is a singular point.

x_1 and x_3 are critical point.

Thm: Local extreme values

If the function f is defined on the interval I and has a local max or local min val at point $x = x_0$ in I $\Rightarrow$ x_0 must be either critical point of f
singular " "
or endpoint "

Finding Absolute Extreme Values:

Ex1 Find the max and min values of

$$f(x) = x^3 - 3x^2 - 9x + 2 \quad \text{on} \quad [-2, 2]$$

Soln:

singular point:
1.) f is poly function $\Rightarrow$ it is diff so it has no singular point.

2.) critical points:

$f'(x) = 0$

$$3x^2 - 6x - 9 = 0$$
$$3(x^2 - 3x - 3) = 0$$
$$(x-3)(x+1) = 0$$
$$x = 3, \quad x = -1$$

$3 \notin [-2, 2]$ but $x = -1 \in [-2, 2]$.

f has critical point at $x = -1$

3.) End points:

$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 2 = 0$
$f(2) = 2^3 - 3(2)^2 - 9(2) + 2 = -20$
$f(-1) = -1 - 3 + 9 + 2 = 7$

f has abs min -20 at $x = 2$

f has abs. max 7 at $x = -1$

Ex 2   Find the max and min values of

$$f(x) = 3x^{2/3} - 2x \quad \text{on} \quad [-1,1]$$

Soln:

$$f'(x) = 2x^{-1/3} - 2 = \frac{2}{x^{1/3}} - 2 = \frac{2-2x^{1/3}}{x^{1/3}} \qquad x \neq 0.$$

Singular point: f is not differentiable at $x=0$

So f has singular point at $\boxed{x=0}$.

critical point: $f'(x)=0$.

$$\frac{2-2x^{1/3}}{x^{1/3}} = 0 \Rightarrow 2-2x^{1/3} = 0$$

$$x^{1/3} = 1 \Rightarrow \boxed{x=1}$$

is critical point.

end points $x=-1$, $x=1$

$f(-1) = 3+2 = 5$

$f(1) = 3-2 = 1$

$f(0) = 0$

f has a abs max at $x=-1$

" " abs min " $x=0$.

ar

Ex: $f(x) = \frac{1}{x-1}$ on $[2,4]$

Soln: $f(x) = (x-1)^{-1}$

$f'(x) = -(x-1)^{-2} = \frac{-1}{(x-1)^2}$ $x \neq 1$

Singular point: f' is not diff at $x=1$

but $1 \notin$ dom f $\Rightarrow$ it has no singular point at $x=1$

critical point: $f'(x) = \frac{-1}{(x-1)^2} = 0$ not possible.

end-points: $f(2) = \frac{1}{2-1} = 1$ abs max at $x=2$

$f(4) = \frac{1}{4-1} = \frac{1}{3}$ abs min at $x=4$.

Thm: The First derivative Test:

Part I Testing critical points and singular points.

Suppose that f is cont- at x_0 and x_0 is not an end-point of the dom f

(a-) if $\exists$ an open interval (a,b) containing x_0 s.t.

$f'(x) > 0$ on (a, x_0) and $f'(x) < 0$ on (x_0, b)

then f has a local max val at x_0.

(b.) if $\exists$ (a,b) containg x_0 s.t $f'(x) < 0$ on (a, x_0) and $f'(x) > 0$ on (x_0, b)

$\Rightarrow$ f has a local min at x_0.

<u>Part II</u> Testing endpoints of domain.

Suppose x_0 is a left endpoint of the dom f and f is right cont at x_0

(c.) If $f'(x) > 0$ on (x_0, b) $\Rightarrow$ f has a <u>local min</u> at x_0

(d.) if $f'(x) < 0$ " $\Rightarrow$ f " <u>local max</u> at x_0

Suppose x_0 is the right end-point of the domain of f and f is left cont at x_0.

(e.) if $f'(x) > 0$ on (a, x_0) $\Rightarrow$ f has a <u>local max</u> at x_0

(f.) if $f'(x) < 0$ " (a, x_0) $\Rightarrow$ f has a <u>local min</u> at x_0

<u>Remark:</u> If f' is post or negative for $\forall\, x \in D_f$

$\Rightarrow$ f has neither max nor a min val at any point