$$x_i = 0 + \Delta x_i = 0 + \frac{2}{n} \cdot i$$

$$f(x_i) = x_i + 1 = \frac{2i}{n} + 1$$

i'th subinterval $\left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \Rightarrow \Delta x_i = \frac{2i}{n} - \frac{2i}{n} + \frac{2}{n}$

$\downarrow$ x_{i-1} $\quad\downarrow$ x_i

$$\boxed{\Delta x_i = \frac{2}{n}}$$

$n \to \infty \quad \Delta x_i \to 0$.

$$S_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

$$= \sum_{i=1}^{n} \left(\frac{2i}{n} + 1\right) \frac{2}{n}$$

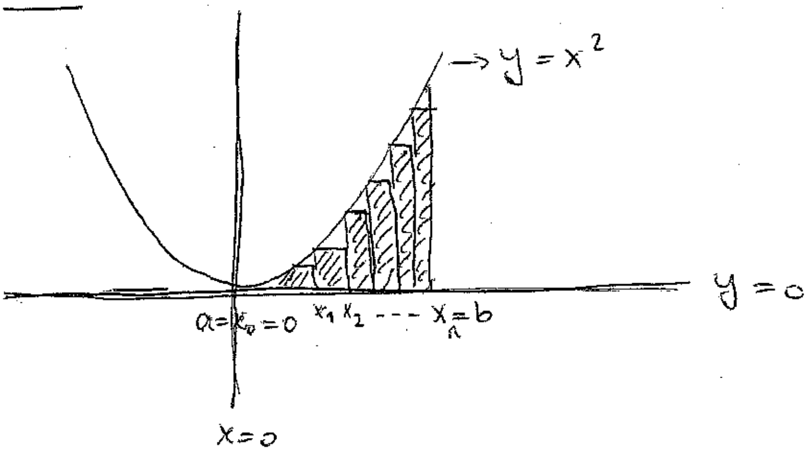
$$= \frac{2}{n}\left[\frac{2}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right]$$

$$= \frac{2}{n}\left[\frac{2}{n}\left(\frac{n(n+1)}{2}\right) + n\right] = \frac{2}{n}\left[2n+1\right] = 4 + \frac{2}{n}.$$

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(4 + \frac{2}{n}\right) = 4 \,//.$$

<u>Ex:</u> Find the area of the region bounded by the parabol $y = x^2$ and the straight lines $y=0$, $x=0$ and $x=b$, $b>0$

<u>Soln:</u>



$y = f(x) = x^2 \quad [0, b]$

$$\Delta x_i = \frac{b-0}{n} = \frac{b}{n}$$

$$x_0 = 0 + 0 \cdot \frac{b}{n} = 0$$

$$x_1 = 0 + 1 \cdot \frac{b}{n} = \frac{b}{n}$$

$\vdots$

$$x_n = 0 + n \cdot \frac{b}{n} = b.$$

$x_i = a + i\Delta x = 0 + i \cdot \frac{b}{n} = \frac{ib}{n}$. $\qquad f(x_i) = \left(\frac{ib}{n}\right)^2$

$$S_n = \sum f(x_i)\,\Delta x_i = \sum \left(\frac{i^2 b^2}{n^2}\right) \cdot \frac{b}{n}$$

$$S_n = \frac{b^3}{n^3} \sum i^2 = \frac{b^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{b^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{b^3}{6}\left[\frac{2n^2+3n+1}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{b^3}{6}\left[\frac{\cancel{n^2}\left(2+\cancel{\frac{3}{n}}^{0}+\cancel{\frac{1}{n^2}}^{0}\right)}{\cancel{n^2}}\right] = \frac{b^3}{3} \ //.$$

NO

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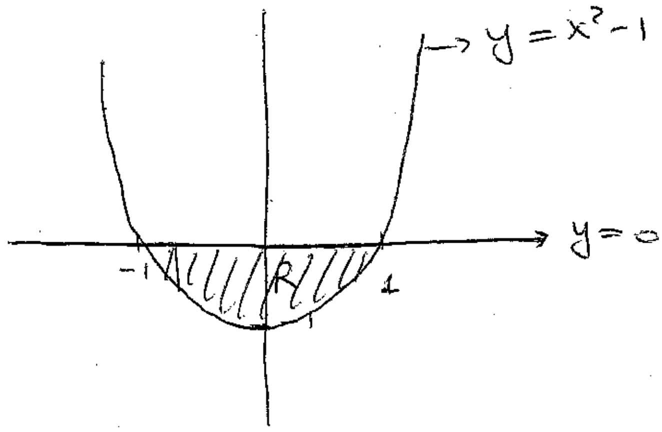
Ex: Find the area of the region above $y = x^2 - 1$ and below $y = 0$ (use equal length subintervals).

Soln:



$f(x) = x^2 - 1 \qquad [-1, 1]$.

$\Delta x_i = \frac{1-(-1)}{n} = \frac{2}{n} \qquad x_i = -1 + i\frac{2}{n} = \frac{-n+2i}{n}$.

$$S_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i \qquad f(x_i) = \left(\frac{-n+2i}{n}\right)^2 - 1$$

$$= \frac{\cancel{n^2} - 4ni + 4i^2 - \cancel{n^2}}{n^2}$$

$$= \frac{-4ni + 4i^2}{n^2}$$

$$= \sum_{i=1}^{n} \left(\frac{-4ni + 4i^2}{n^2}\right) \cdot \frac{2}{n}$$

$$= \frac{-8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{-8}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= -\frac{4}{n^2}\left[n^2 + n\right] + \frac{4}{3n^2}\left[2n^2 + 3n + 1\right]$$

$$= -4 - \cancel{\frac{4}{n}} + \frac{8}{3} + \cancel{\frac{4}{n}} + \frac{4}{3n^2}$$

$$= -\frac{4}{3} + \frac{4}{3} \cdot \frac{1}{n^2} = \frac{4}{3}\left[-1 + \frac{1}{n^2}\right]$$

$$A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{4}{3}\left[-1 + \frac{1}{n^2}\right] = -\frac{4}{3}$$

5.3 The Definite Integral

Partitions and Riemann Sums:

Let P be a finite set of points arranged in order between a and b.

$$P = \{ x_0, x_1, \ldots x_{n-1}, x_n \}$$

where $a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$. P is called partition of $[a,b]$. It divides $[a,b]$ into n subintervals.

Since f is continuous on each subinterval $[x_{i-1}, x_i]$ of it, it takes on max and min values at points of that interval.

Thus $\exists$ l_i, u_i in $[x_{i-1}, x_i]$ s.t.

$$f(l_i) \leq f(x) \leq f(u_i) \qquad x_{i-1} \leq x \leq x_i$$

Defn:

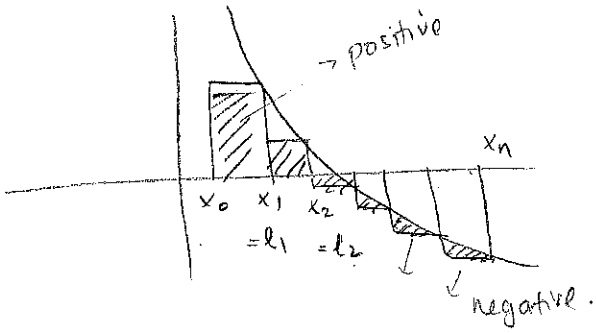
The lower (Riemann) sum, $L(f,P)$ and the upper (Riemann) sum $U(f,P)$ for function f and the partition P

$$L(f,P) = f(l_1)\Delta x_1 + f(l_2)\Delta x_2 + \ldots + f(l_n)\Delta x_n$$

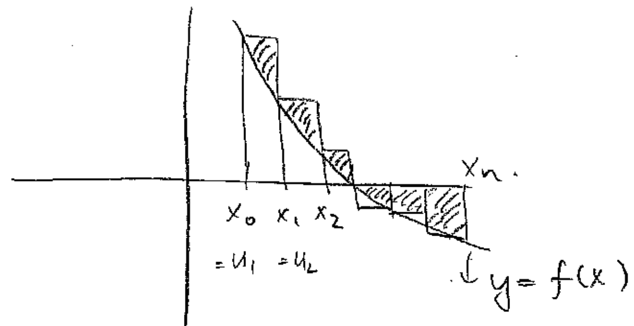
$$L(f,P) = \sum_{i=1}^{n} f(l_i)\Delta x_i$$

$$U(f,P) = f(u_1)\Delta x_1 + f(u_2)\Delta x_2 + \ldots + f(u_n)\Delta x_n$$

$$U(f,P) = \sum_{i=1}^{n} f(u_i)\Delta x_i$$



Lower Riemam sum



Upper Riemann Sum

Ex: Calculate lower and upper Riemann sums for $f(x)=\frac{1}{x}$ on $[1,2]$ corresponding to the partition P of $[1,2]$ into 4 subintervals of equal length.

Soln:

$$\Delta x_i = \frac{2-1}{4} = \frac{1}{4} \qquad x_i = a + i\Delta x$$

$$x_0 = 1 \qquad x_3 = 1+3\cdot\frac{1}{4} = \frac{7}{4}$$

$$x_1 = 1 + 1\cdot\frac{1}{4} = \frac{5}{4} \qquad x_4 = 1+4\cdot\frac{1}{4} = 2$$

$$x_2 = 1+2\cdot\frac{1}{4} = \frac{3}{2}$$

f is decreasing on $[1,2]$ $\Rightarrow$ its min val and max val on ith interval $[x_{i-1}, x_i]$ is $\frac{1}{x_i}$ and $\frac{1}{x_{i-1}}$

$$L(f,P) = \Delta x \cdot \sum_{i=1}^{4} f(l_i)$$
$$= \Delta x \left[f(l_1) + f(l_2) + \dots + f(l_4) \right]$$
$$= \Delta x \left[\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_4} \right]$$
$$= \frac{1}{4}\left[\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2} \right] \cong 0.6345$$

$$U(f,P) = \Delta x \sum_{i=1}^{n} f(u_i) = \Delta x \left[f(u_1) + f(u_2) + f(u_3) + f(u_4) \right]$$

$$= \Delta x \left[\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \right]$$

$$U(f,P) = \frac{1}{4} \left[1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} \right] \approx 0.7595$$

Ex: Calculate the lower and upper Riemann sums for $f(x) = x^2$ on $[0, a]$ $(a > 0)$ corresponding to the partition P_n of $[0,a]$ into n subinterva of equal length.

Soln: $\Delta x_i = \frac{a-0}{n} = \frac{a}{n}$ $\qquad x_i = 0 + i\frac{a}{n} = \frac{ai}{n}$ $\quad i = 0, 1, 2 \dots n.$

$f(x) = x^2$ is increasing on $[0, a]$

max and min val of f on ith sub interval $[x_{i-1}, x_i]$ occur at $\boxed{l_i = x_{i-1}}$ and $\boxed{u_i = x_i}$

$$L(f,P_n) = \sum_{i=1}^{n} f(l_i)\Delta x = \sum_{i=1}^{n} (x_{i-1})^2 \cdot \frac{a}{n} = \sum_{i=1}^{n} \frac{(i-1)^2 \cdot a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2$$

$$= \frac{a^3}{n^3} \left[\frac{(n-1)\, n\, (2(n-1)+1)}{6} \right] = \frac{a^3 (n-1)(2n-1)}{6n^2}$$

Upper Riemann Sum

$$U(f, P_n) = \sum_{i=1}^{n} (x_i)^2 \Delta x$$

$$= \sum_{i=1}^{n} \frac{a^2 i^2}{n^2} \cdot \frac{a}{n} = \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{a^3 (n+1)(2n+1)}{6n^2}$$

The Definite Integral:

Defn: The Definite Integral

Suppose there is exactly one number I s.t for $\forall$ partition P of $[a,b]$

$$L(f,P) \le I \le U(f,P)$$

then we say that f is <u>integrable</u> on $[a,b]$.

$$\boxed{\underbrace{I}_{\text{Definite Integral of } f \text{ on } [a,b].} = \int_a^b f(x)\,dx}$$

$\int_a^b f(x)\,dx$

upper limit of integration.

integral sign

lower limit of integration

f : integrand

x : variable of integration.

dx : differential of x.

Ex: Show that $f(x) = x^2$ is integrable over $[0, a]$ where $a>0$, and evaluate $\int_0^a x^2\,dx$.

Soln:

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{(n-1)(2n-1)a^3}{6n^2} = \lim_{n\to\infty} \frac{a^3 \cdot \cancel{n^2}\left(2 - \cancel{\frac{3}{n}}^{\,0} + \cancel{\frac{1}{n^2}}^{\,0}\right)}{6\cancel{n^2}}$$

$$= \frac{a^3}{3} \,//$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)a^3}{6n^2} = \lim_{n\to\infty} \frac{a^3 \cancel{n^2}\left(2 + \cancel{\frac{3}{n}}^{\,0} + \cancel{\frac{1}{n^2}}^{\,0}\right)}{6\cancel{n^2}}$$

$$= \frac{a^3}{3}.$$

if $L(f,P_n) \leq I \leq U(f,P_n)$

$\Rightarrow \quad I = \dfrac{a^3}{3}$ so, $f(x) = x^2$ is integrable over $[0,a]$.

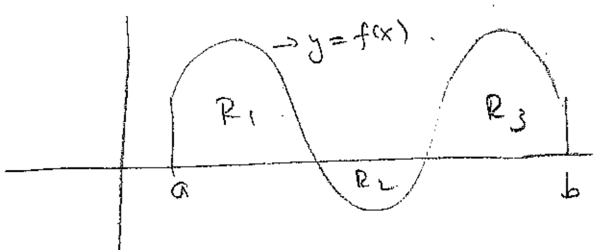
$$\int_0^a f(x)\,dx = \boxed{\int_0^a x^2\,dx = \frac{a^3}{3}} \,//.$$

For all partitions P of $[a,b]$

$$L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P)$$

For general f,

$$\int_a^b f(x)\,dx = \text{area of } R_1 - \text{area of } R_2 + \text{area of } R_3$$



General Riemann Sums

$P = \{x_0, x_1, \dots, x_n\}$ $\quad a = x_0 < x_1 < x_2 \dots < x_n = b$ partition of [a,

$\|P\| = \max\limits_{1 \le i \le n} \Delta x_i$, c_i is a point in $[x_{i-1}, x_i]$.

and let $c = (c_1, c_2, \dots c_n)$

$$\boxed{R(f, P, c) = \sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + \dots + f(c_n)\Delta x_n.}$$

called <u>Riemann Sum</u> of f on $[a,b]$ corresponding to part. P

$$\boxed{\lim_{\substack{n(P) \to \infty \\ \|P\| \to 0}} R(f, P, c) = \int_a^b f(x)\,dx.}$$

<u>Thm</u>: If f is <u>continuous</u> on $[a,b]$ $\Rightarrow$ f is <u>integrable</u> on $[a,b]$

Sec 5.4 Properties of The Definite Integral:

Thm: Let f, g be integrable on an interval containing the points a, b and c.

$\Rightarrow$

(1-) $$\int_a^a f(x)\,dx = 0$$

(2-) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(3) $$\int_a^b \left(Af(x) + Bg(x)\right)dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx$$

(4-) $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

(5-) if $a \le b$, $f(x) \le g(x)$, $a \le x \le b$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

(6.) if $a \le b$

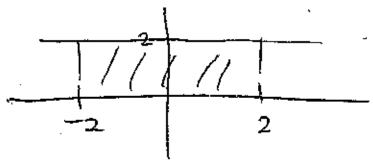
$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$ Triangle Inequality for definite integrals

(7-) $$\int_{-a}^a f(x)\,dx = 0$$ f: odd function.

(8-) $$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$ f: even function.

Ex: Evaluate,

a-) $\displaystyle\int_{-2}^{2}(2+5x)\,dx$



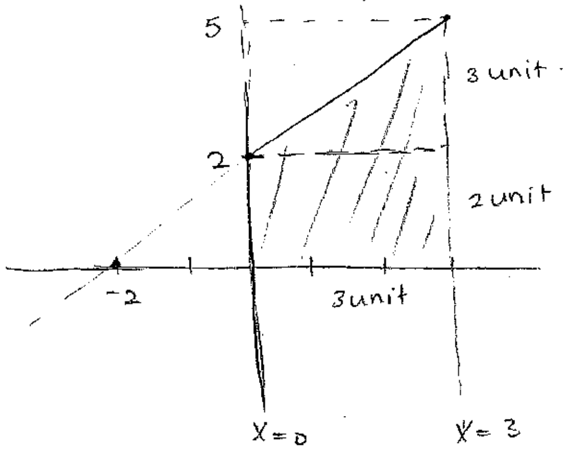
$$\underset{\text{by (3)}}{=}\int_{-2}^{2}2\,dx+5\int_{-2}^{2}x\,dx.$$

$$\int_{-2}^{2}x\,dx=0$$

odd function.

$$=8+0=\boxed{8}$$

b.) $\displaystyle\int_{0}^{3}(2+x)\,dx$.

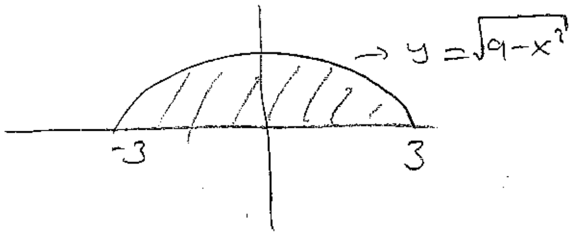


= area of triangle + area of rectangle

$$=\frac{1}{2}(3)(3)+(3)(2)$$

$$=\boxed{\frac{21}{2}}$$

(c.) $\displaystyle\int_{-3}^{3}\sqrt{9-x^2}\,dx$



= area of semicircle

$$=\frac{1}{2}\pi(3^2)=\boxed{\frac{9\pi}{2}}$$

Mean-Value Thm For Integrals

Thm: if f is continuous on $[a,b]$ then $\exists$ a point c in $[a,b]$

s.t $$\boxed{\int_a^b f(x)\,dx = (b-a)\,f(c)}$$

Defn: Average value of a function

if f is integrable on $[a,b]$, then the average value or mean value of f on $[a,b]$

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

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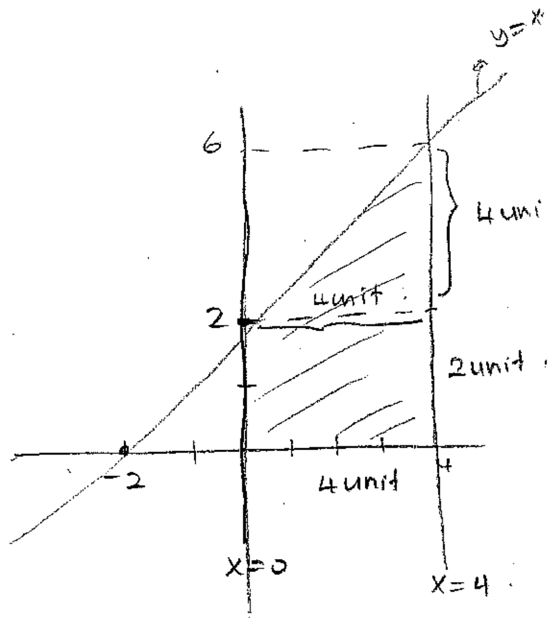
Ex: Find the average value of

$f(x) = x+2$ over $[0,4]$

Soln:
$$\bar{f} = \frac{1}{4-0}\int_0^4 (x+2)\,dx.$$

$$= \frac{1}{4}\left[4(2) + \frac{1}{2}(4)(4)\right]$$

$$= 4 \;//$$



Q32.

Ex: Find average value of $k(x) = x^2$ over $[0,3]$.

Soln.

$$\bar{f} = \frac{1}{3-0} \int_0^3 x^2 \, dx$$

$$\boxed{\int_0^a x^2 \, dx = \frac{a^3}{3}}$$

$$= \frac{1}{3} \cdot \frac{3^3}{3} = \frac{9 \times 3}{9} = 3 \,//.$$

Definite Integrals of Piecewise Continuous Functions:

Defn:

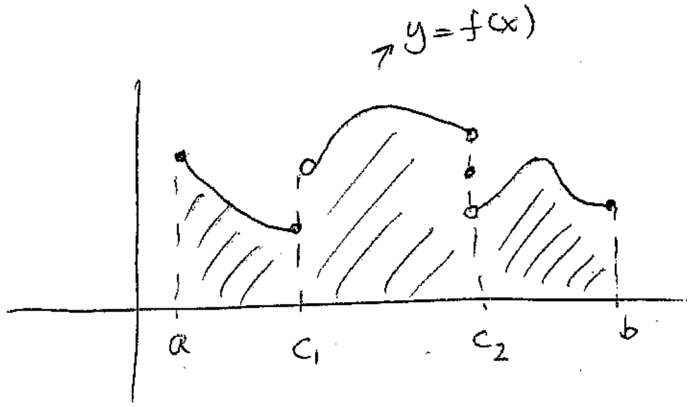
Let $c_0 < c_1 < c_2 \cdots < c_n$ finite set of points.

A function f is defined on $[c_0, c_n]$ except ~~poss~~ at some of the points c_i, $(0 \leq i \leq n)$, is called piecewise continuous on that interval if for each i $(\phi \leq i \leq n)$, $\exists$ function F_i continuous on the closed interval $[c_{i-1}, c_i]$ s.t

$$f(x) = F_i(x) \quad \text{on} \quad (c_{i-1}, c_i)$$

then,

$$\boxed{\int_{c_0}^{c_n} f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} F_i(x)\,dx.}$$



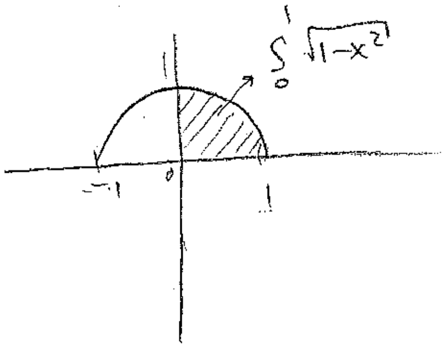
$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{b} f(x)\,dx.$$

<u>Ex:</u> Find $\int_0^3 f(x)\,dx$.

$$\text{where } f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x < 2 \\ x-2 & \text{if } 2 < x < 3 \end{cases}$$

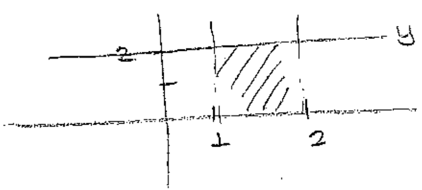
<u>Soln:</u>

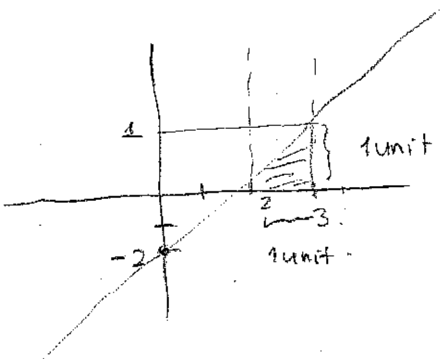
$$\int_0^3 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx + \int_1^2 2\,dx + \int_2^3 (x-2)\,dx.$$



$$= \left(\frac{1}{4}\pi(1)^2\right) + 1\times 2 + \frac{1}{2}(1\times 1)$$

$$= \frac{\pi+10}{4} \quad //.$$





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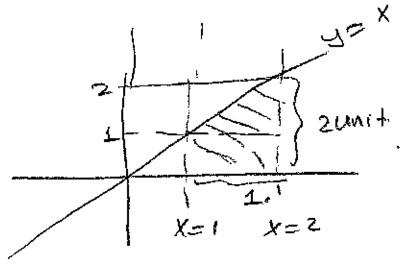
Q37.

<u>Ex:</u> Find $\int_0^2 g(x)\,dx$ $\quad g(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2 \end{cases}$

<u>Soln:</u> $\int_0^2 g(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x\,dx$.



$$= \frac{1^3}{3} + \frac{1 \times \cancel{1}}{\cancel{2}} + 1 \times 1$$

$$= \frac{1}{3} + \frac{1}{2} + 1 = \frac{2}{6} + \frac{3}{6} + \frac{6}{6} = \frac{11}{6} //$$

Q38.

<u>Ex!</u> Evaluate $\int_{-1}^{2} \operatorname{Sgn} x\,dx$, where.

<u>Soln:</u>

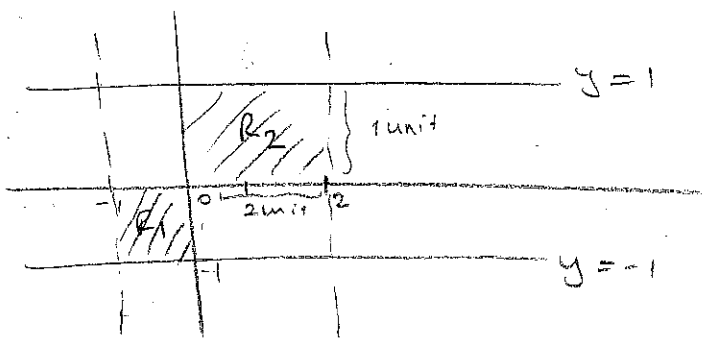
$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

$$\int_{-1}^{2} \operatorname{sgn} x\,dx = \int_0^2 1\,dx - \int_{-1}^{0} -1\,dx$$

$$= (2 \times 1) - (-1)(-1)$$

$$= 2 - 1$$

$$= 1 //$$



Sec 5.5 The Fundamental Thm of Calculus

Thm: Suppose f is continuous on interval I containing point a

Part 1 Let F defined on I.

~~best~~ $$\boxed{F(x) = \int_a^x f(t)\,dt}$$

$\Rightarrow$ F is differentiable on I, $F'(x) = f(x)$ there.

thus $F'(x) = \boxed{\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dx = f(x)}$. F is antiderivative of f on I

Part 2 If $G(x)$ is any antiderivative of $f(x)$ on I,

so $G'(x) = f(x)$ on I then for any b in I

$$\boxed{\int_a^b f(x)\,dx = G(b) - G(a)}$$

Defn: $$F(x)\Big|_a^b = F(b) - F(a).$$

thus, $$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

$$\frac{d}{dx}\frac{x^3}{3} = x^2$$

Ex: 1 Evaluate

a-) $$\int_0^a x^2\,dx = \frac{x^3}{3}\Big|_0^a = \frac{a^3}{3} - \frac{0^3}{3} = \frac{a^3}{3}$$

b.) $$\int_{-1}^{2} (x^2 - 3x + 2)\,dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Big|_{-1}^{2}$$

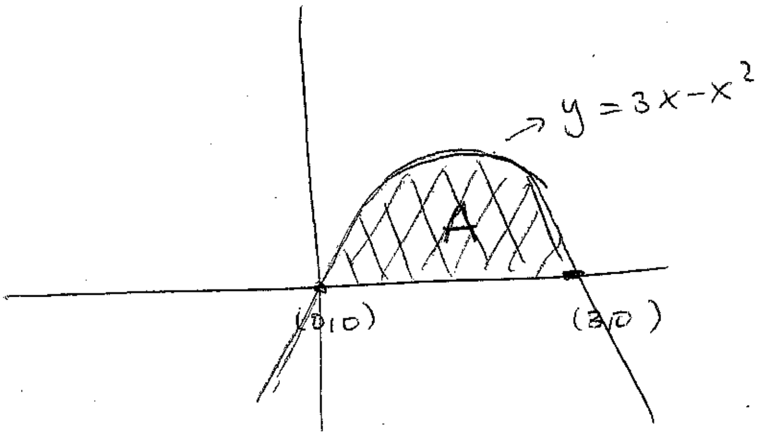
$$= \left[\frac{2^3}{3} - \frac{3\cdot 2^2}{2} + 2(2)\right] - \left[\frac{(-1)^3}{3} - \frac{3(-1)^2}{2} + 2(-1)\right]$$

$$= \frac{9}{2}$$

Ex2 Find the area A of the plane region lying above the x-axis and under the curve $y = 3x - x^2$

Soln: $y = 3x - x^2$

$y = 0 \Rightarrow \quad 0 = 3x - x^2 = x(3-x)$ $\quad x = 0$, $x = 3$.



$$A = \int_0^3 3x - x^2 = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \left(\frac{3\cdot 3^2}{2} - \frac{3^3}{3}\right) - (0 - 0) = \frac{9}{2}$$

<u>Ex3</u> Find the average value of

$$f(x) = e^{-x} + \cos x \quad \text{on } [-\tfrac{\pi}{2}, 0]$$

<u>Soln:</u>

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx .$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

$$\bar{f} = \frac{1}{0-(-\frac{\pi}{2})} \int_{-\frac{\pi}{2}}^{0} (e^{-x} + \cos x)\,dx$$

$$= \frac{2}{\pi} \left(-e^{-x} + \sin x \right|_{(-\frac{\pi}{2})}^{0}$$

$$= \frac{2}{\pi}\left[(-e^{-0} + \sin 0) - (-e^{\frac{\pi}{2}} + \sin(-\tfrac{\pi}{2})) \right]$$

$$= \frac{2}{\pi}\left[\cancel{-1} + 0 + e^{\pi/2} + \cancel{1} \right] = \frac{2}{\pi} e^{\pi/2} .$$

<u>Remark!</u> if we build chain Rule to 1st conclusion of Fundamental thm,

$$\boxed{\begin{aligned} \frac{d}{dx} \int_a^{g(x)} f(t)\,dt &= f(g(x)) \cdot g'(x) \\ \frac{d}{dx} \int_{h(x)}^{g(x)} f(t)\,dt &= f(g(x))\, g'(x) - f(h(x)) \cdot h'(x) \end{aligned}}$$

Ex: Find the derivatives of

$$F(x) = \int_a^x f(t)\,dt .$$

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(a-) $$F(x) = \int_x^3 e^{-t^2}\,dx .$$

Soln:
$$F'(x) = \frac{d}{dx}\int_x^3 e^{-t^2}\,dx .$$

$$= \frac{d}{dx}\left[-\int_3^x e^{-t^2}\,dx\right]$$

$$F'(x) = -e^{-x^2}$$

(b.) $$G(x) = x^2\int_{-4}^{5x} e^{-t^2}\,dt .$$

Soln:
$$G'(x) = 2x\int_{-4}^{5x} e^{-t^2}\,dt + x^2 \cdot \frac{d}{dx}\int_{-4}^{5x} e^{-t^2}\,dt$$

$$= 2x\int_{-4}^{5x} e^{-t^2}\,dt + x^2 \cdot e^{-(5x)^2} \cdot 5 .$$

$$G'(x) = 2x\int_{-4}^{5x} e^{-t^2}\,dt + 5x^2 e^{-25x^2}$$

(c.) $$H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt .$$

soln:
$$H'(x) = e^{-(x^3)^2}(3x^2) - e^{-(x^2)^2} \cdot 2x$$

$$= 3x^2 e^{-x^6} - 2x\,e^{-x^4}$$

//

Ex: $\displaystyle\int_{\pi/4}^{\pi/3} \sin\theta \, d\theta = -\cos\theta \Big|_{\frac{\pi}{4}}^{\pi/3}$

$= -\cos\frac{\pi}{3} + \cos\frac{\pi}{4} \quad = -\frac{\sqrt{2}}{2} + \frac{1}{2}$

$= -$

Sec 5.6 The Method of Substitution

Elementary Integrals:

(1.) $\int 1 \, dx = x + C$

(2.) $\int x \, dx = \frac{1}{2}x^2 + C$

(3.) $\int x^2 \, dx = \frac{x^3}{3} + C$

(4.) $\int \frac{1}{x^2} \, dx = \frac{-1}{x} + C$

(5.) $\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C$

(6.) $\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + C$

(7.) $\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C$

(8.) $\int \frac{1}{x} \, dx = \ln|x| + C$

(9.) $\int \sin ax \, dx = \frac{-1}{a} \cos ax + C$

(10.) $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$

(11.) $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$

(12.) $\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C$

(13.) $\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C$

(14.) $\int \csc ax \cot ax \, dx = -\frac{1}{a} \csc ax$

(15.) $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\frac{x}{a} + C \quad a > 0$

(16.) $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$

(17) $\int e^{ax}dx = \frac{1}{a}e^{ax}+c$ (18) $\int b^{ax}dx = \frac{b^{ax}}{a\ln b}+c$

(19) $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + c$ (20) $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + c$

Ex1

a) $\int (x^4-3x^3+8x^2-6x-7)\,dx = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{8x^3}{3} - \frac{6x^2}{2} - 7x + C$

b) $\int \left(5x^{3/5} - \frac{3}{2+x^2}\right)dx = 5\frac{5}{8}x^{8/5} - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$

$= \frac{25}{8}x^{8/5} - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$

c) $\int \left(\frac{1}{\pi x} + a^{\pi x}\right)dx = \frac{1}{\pi}\ln|x| + \frac{a^{\pi x}}{\pi\ln a} + c$ $a>0$.

d) $\int (4\cos 5x - 5\sin 3x)\,dx = 4\cdot\frac{1}{5}\sin 5x + 5\cdot\frac{1}{3}\cos 3x$

$= \frac{4}{5}\sin 5x + \frac{5}{3}\cos 3x$

Ex: $\int \frac{(x+1)^2}{x}dx = \int \frac{x^2+2x+1}{x}dx$

$= \int \left(x+2+\frac{1}{x}\right)dx = \frac{x^2}{2}+2x+\ln|x|+c$.

Method of Substitution : (integral version of chain Rule)

By chain rule, $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$.

then, $\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$.

Let $u = g(x)$.

$$\int f'(u)\, du = f(u) + C = f(g(x)) + C.$$

$$\frac{du}{dx} = g'(x)$$

$$\Rightarrow \quad du = g'(x)\, dx$$

Ex 1 $\int \frac{x}{x^2+1}\, dx$

Soln: Let $u = x^2+1$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$.

$$\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2+1) + c.$$

$$= \ln\sqrt{x^2+1} + c.$$

Ex 2 $\int \frac{\sin(3\ln x)}{x}\, dx$

$u = 3\ln x$

$du = 3 \cdot \frac{1}{x}\, dx$.

$\frac{1}{3} du = \frac{1}{x} dx$

$$= 3\int \sin u\, du$$

$$= -\frac{1}{3}\cos u + C$$

$$= -\frac{1}{3}\cos(3\ln x) + C.$$

Ex3 $\int e^x \sqrt{1+e^x}\, dx$

$u = 1+e^x$

$du = e^x dx$

$$= \int \sqrt{u}\, du = \int u^{1/2} du = \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (1+e^x)^{3/2} + C.$$

Ex4 $\int \frac{1}{x^2+4x+5}\, dx$

Soln: $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \int \frac{du}{u^2+1}$

Let $u = x+2$

$du = dx.$

$$= \tan^{-1} u + C$$

$$= \tan^{-1}(x+2) + C$$

Ex5 $\int \frac{dx}{\sqrt{e^{2x}-1}}$

$u = e^{-x}$

$du = -e^{-x} dx$

$-du = e^{-x} dx$

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{e^{2x}(1-e^{-2x})}} = \int \frac{dx}{e^x \sqrt{1-(e^{-x})^2}}$$

$$= -\int \frac{du}{\sqrt{1-u^2}} = -\sin^{-1} u + C = -\sin^{-1}(e^{-x}) + C$$

Thm: Substitution in a definite Integral

Suppose that g is differentiable function on $[a,b]$ and $g(a) = A$ and $g(b) = B$. Also suppose that f is continuous on the range of g. Then

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_A^B f(u)\, du.}$$

Ex1 Evaluate the integral $I = \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\, dx.$

Soln: $u = \sqrt{x+1}$ if $x = 8 \Rightarrow u = \sqrt{9} = 3$

$du = \frac{dx}{2\sqrt{x+1}}$ if $x = 0 \Rightarrow u = \sqrt{1} = 1$

$$I = 2\int_1^3 \cos u\, du = 2\sin u \Big|_1^3 = 2\sin 3 - 2\sin 1.$$

method 2:

$$I = \int_{x=0}^{x=8} \cos u\, du = 2\sin u \Big|_{x=0}^{x=8} = 2\sin\sqrt{x+1}\Big|_{x=0}^{x=8} = 2\sin 3 - 2\sin 1$$

Ex2 Find the area of the region bounded by

$y = \left(2 + \sin\frac{x}{2}\right)^2 \cos\frac{x}{2}$, $y = 0$, $x = 0$ and $x = \pi$

Soln: $A = \int_0^\pi \left(2 + \sin\frac{x}{2}\right)^2 \cos\frac{x}{2}\, dx =$

$y \geq 0$ when $0 \leq x \leq \pi$

$u = 2 + \sin\frac{x}{2}$ $\qquad$ $x = \pi \Rightarrow u = 2 + \sin\frac{\pi}{2} = 3$

$du = \frac{1}{2}\cos\frac{x}{2}\,dx$ $\qquad$ $x = 0 \Rightarrow u = 2 + \sin 0 = 2$

$2\,du = \cos\frac{x}{2}\,dx.$

$$A = 2\int_2^3 u^2\,du = \frac{2}{3}u^3\Big|_2^3 = \frac{2}{3}\left[3^3 - 2^3\right] = \frac{38}{3} \text{ square units}$$

Trigonometric Integrals:

1.) $\int \tan x\,dx = \ln|\sec x| + C$

2.) $\int \cot x\,dx = \ln|\sin x| + C = -\ln|\csc x| + C$

3.) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

4.) $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

$= \ln|\csc x - \cot x| + C.$

Consider integrals of the form

$\int \sin^m \cos^n x\,dx.$ $\rightarrow$ if either m or n is <u>odd positive</u> integer

if $\boxed{n = 2k+1}$ $\qquad$ $\sin^2 x + \cos^2 x = 1$

$$\int \sin^m (\cos^2)^k \cdot \cos x\,dx = \int \sin^m (1 - \sin^2 x)^k \cos x\,dx.$$

Make $\boxed{u = \sin x}$

Similarly if $\boxed{m = 2k+1}$ make substitution. $\boxed{u = \cos x}$

Ex: Evaluate

(a.) $\int \sin^3 x \cos^8 x \, dx$

$$= \int (1-\cos^2 x) \cos^8 x \sin x \, dx$$

Let $u = \cos x$
$du = -\sin x \, dx$.

$$= -\int (1-u^2) u^8 \, du$$

Ex: $\int \sin^5 x \, dx$

$$= \int (u^{10} - u^8) \, du = \frac{u^{11}}{11} + \frac{u^9}{9} + C$$

$$= \frac{\cos^{11} x}{11} - \frac{\cos^9 x}{9} + C$$

(b.) $\int \cos^5 ax \, dx =$

$u = \sin ax$

$du = a \cos ax \, dx$

$$= \int (1-\sin^2 ax)^2 . \cos ax \, dx$$

$$= \frac{1}{a} \int (1-u^2)^2 \, du$$

$$= \frac{1}{a} \int (1 - 2u^2 + u^4) \, du = \frac{1}{a} \left(u - \frac{2u^3}{3} + \frac{u^5}{5} \right) + C$$

$$= \frac{1}{a} \left(\sin ax - \frac{2}{3} \sin^3 ax + \frac{1}{5} \sin^5 ax \right) + C$$

Note: If the powers of $\sin x$ and $\cos x$ are both even

$\Rightarrow$ we use double angle formulas.

$$\boxed{\begin{aligned} \cos^2 x &= \frac{1}{2}(1+\cos 2x) \\ \sin^2 x &= \frac{1}{2}(1-\cos 2x) \end{aligned}}$$

Ex: $\int \cos^2 x \, dx$

Soln: $= \frac{1}{2}\int (1+\cos 2x)\,dx$ $\qquad \sin 2x = 2\sin x \cos x$

$= \frac{1}{2}\left[\int 1\,dx + \int \cos 2x\,dx\right]$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

$= \frac{x}{2} + \frac{1}{2}\sin x \cos x + C$

Exercise: $\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C$

Soln: $= \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C = \frac{1}{2}x - \frac{1}{4}\cdot 2\sin x \cos x + C$

$= \frac{1}{2}(x - \sin x \cos x) + C$.

Ex: Evaluate $\int \sin^4 x \, dx$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Soln:

$= \frac{1}{4}\int (1-\cos 2x)^2\,dx$ $\qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$ $\qquad \cos^2 2x = \frac{1}{2}(1+\cos 4x)$

$= \frac{1}{4}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx$

$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$.

<u>Note:</u> To calculate

$\int \sec^m x \tan^n x \, dx$ or $\int \csc^m x \cot^n x \, dx$

we use
$\sec^2 x = 1 + \tan^2 x$ and
$\csc^2 x = 1 + \cot^2 x$ and

one of the substitution $u = \sec x$, $u = \tan x$, $u = \csc x$, $u = \cot x$

<u>Remark:</u> if m is <u>odd</u> and n is <u>even</u> then in that case we use the method <u>integration by parts</u>.

<u>Ex:</u> Evaluate, $(\tan x)' = \sec^2 x$

a.) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx =$

$= \int \sec^2 x \, dx - \int dx$

$= \tan x - x + C$.

b.) $\int \sec^4 t \, dt = \int (1 + \tan^2 t) \cdot \sec^2 t \, dt$

$u = \tan t$ $\qquad = \int (1 + u^2) \, du$

$du = \sec^2 t \, dt$ $\qquad = u + \frac{1}{3} u^3 + C$

$= \tan t + \frac{1}{3} \tan^3 t + C$.

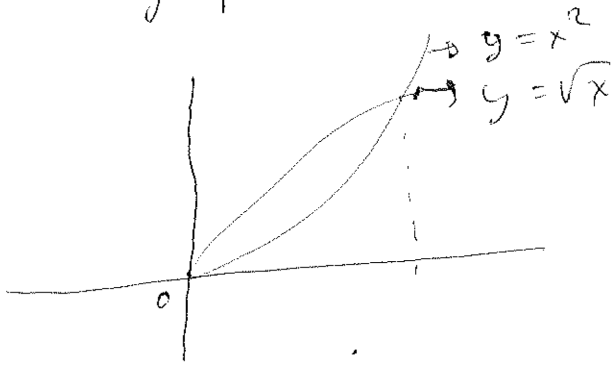
c.) $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \, (\sec^2 x - 1) \sec x \tan x \, dx$

$u = \sec x$ $\qquad = \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$du = \sec x \tan x \, dx$

$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$ //.

Ex1) Find the area of the region bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$



To find intersection

$x^2 = \sqrt{x}$

$x^4 = x \Rightarrow x^4 - x = 0$

$x(x^3 - 1) = 0$

$x(x-1)(x^2+x+1) = 0$

$x = 0, \; x = 1$

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx$$

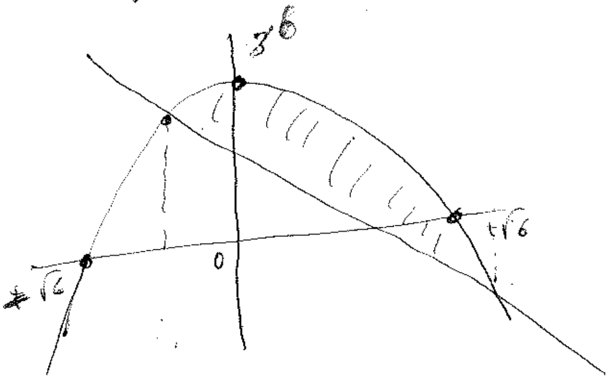
$$= \int_0^1 (x^{1/2} - x^2)\,dx = \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ unit square}$$

Ex2) Find the area of the region bounded by the graphs of $y = 6 - x^2$ and $y = 3 - 2x$

$y = 3 \Rightarrow (0, 3)$

$y = 0 \Rightarrow (3/2, 0)$



To find intersection

$6 - x^2 = 3 - 2x$

$\Rightarrow x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, \; x = -1$

$$A = \int_{-1}^{3} \left((6 - x^2) - (3 - 2x)\right)dx$$

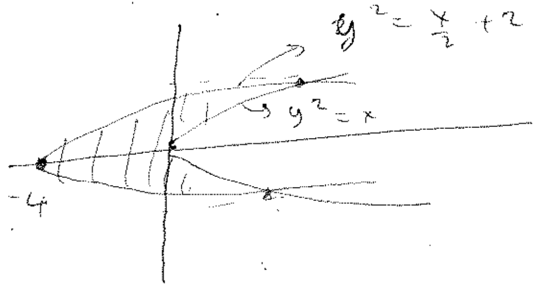
$$= \int_{-1}^{3} (3 + 2x - x^2)\,dx = 3x + x^2 - \frac{x^3}{3}\Big|_{-1}^{3}$$

$$-2 + \frac{1}{3} = -\frac{5}{3}$$

$$= (9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right)$$

$$= 9 + \frac{5}{3} = \frac{32}{3} \text{ unit square.}$$

Ex 3) Find the area of the region bounded by the curves $2y^2 = x + 4$ and $y^2 = x$



$\cancel{\frac{x}{2} + 4 = x}$

$2y^2 - 4 = y^2$

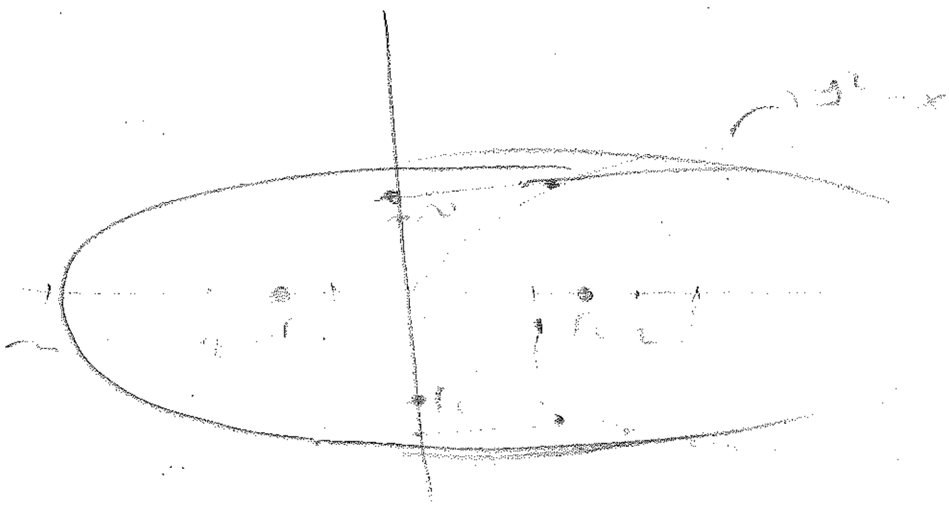
$y^2 = 4 \Rightarrow y = \pm 2$

$$A = \int_{-2}^{2} (y^2 - (2y^2 - 4))\, dy = \int_{-2}^{2} (-y^2 + 4)\, dy = -\frac{y^3}{3} + 4y \Big|_{-2}^{2}$$

$$= -\frac{8}{3} + 8 - \frac{8}{3} + 8$$

$$= 16 - \frac{16}{3} = \frac{32}{3} \text{ unit square}$$

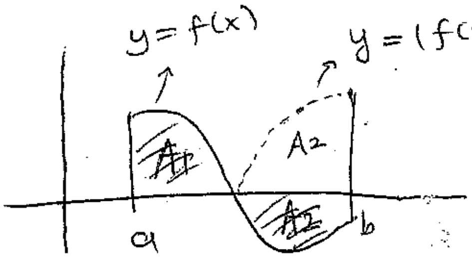
1)



$2y^2 = x + 4$

$y^2 = \frac{x}{2} + 2$

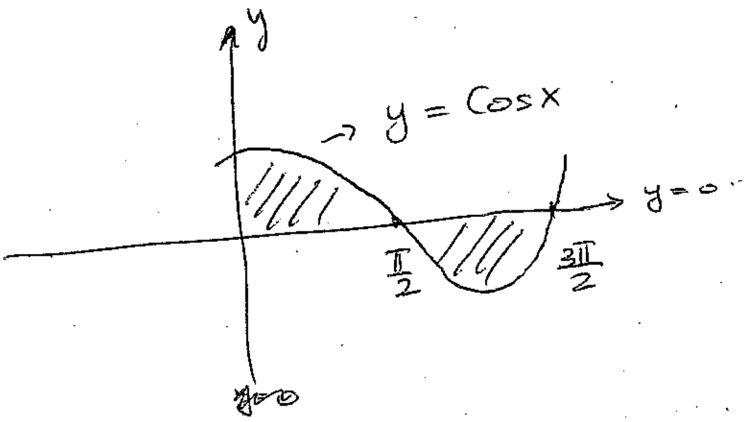
Sec 5.7 Areas of Plane Regions



$$A = A_1 + A_2 .$$

$$\int_a^b f(x)\,dx = A_1 - A_2 \quad \text{and} \quad \int_a^b |f(x)|\,dx = A_1 + A_2 = A .$$

Ex: Find the area bounded by $y = \cos x$, $y = 0$, $x = 0$ and $x = \frac{3\pi}{2}$



$$A = \int_0^{\frac{3\pi}{2}} |\cos x|\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} -\cos x\,dx$$

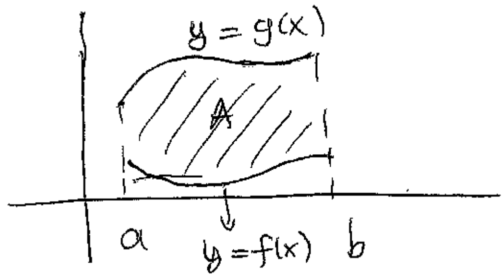
$$= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0 - \sin\frac{3\pi}{2} + \sin\frac{\pi}{2}$$

$$= 1 - 0 + 1 + 1$$

$A = \underline{\underline{3}}$ square units.

Areas between Two Curves



$$A = \int_a^b (g(x) - f(x))\,dx$$

$$A = \int_a^b |f(x) - g(x)|\,dx$$

<u>Ex:</u> Find the area of the bounded plane R lying between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

<u>Soln:</u>

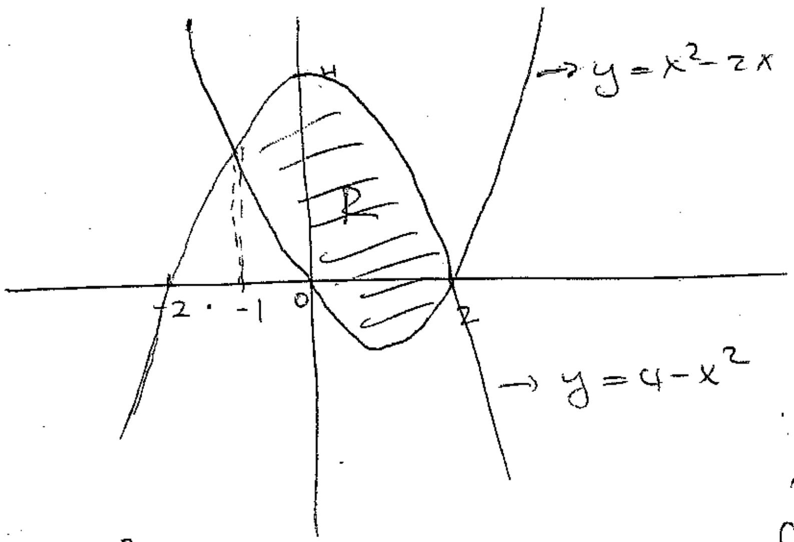
To find intersection of curves:

$$x^2 - 2x = 4 - x^2$$

$$2x^2 - 2x - 4 = 0$$

$$2(x^2 - x - 2) = 2(x-2)(x+1) = 0$$

$$x = 2 \quad x = -1$$

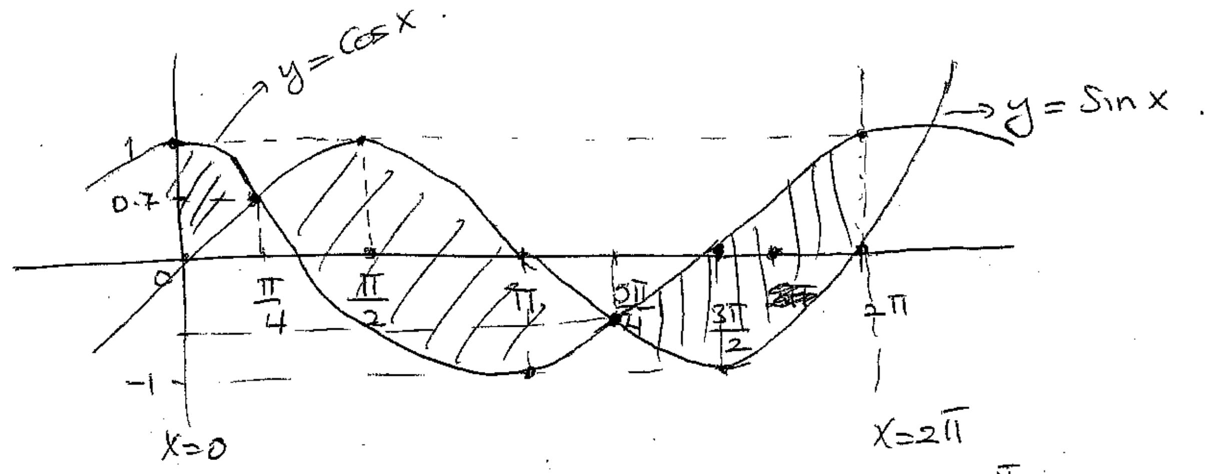


$$A = \int_{-1}^{2} (4 - x^2) - (x^2 - 2x)\,dx = \int_{-1}^{2} (4 - 2x^2 + 2x)\,dx$$

$$= 4x - \frac{2x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} \Big|_{-1}^{2} = 8 - \frac{2}{3}\cdot 8 + 4 - \left[(-4) - \frac{2}{3}(-1)^3 + (-1)^2\right]$$

$$= 9$$

Ex: Find the total area A lying between the curves $y = \sin x$ and $y = \cos x$ from $x=0$ to $x = 2\pi$

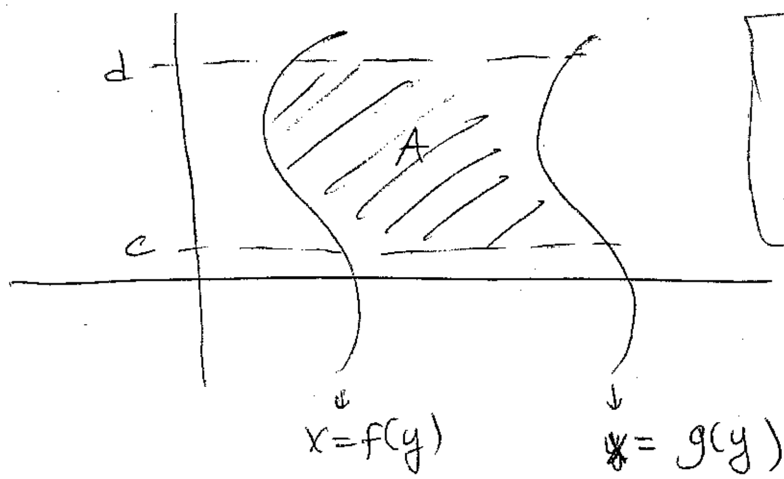


$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{5\pi/4} (\sin x - \cos x)\, dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\, dx$$

$$= (\sin x + \cos x \Big|_0^{\frac{\pi}{4}} - (\cos x + \sin x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + (\sin x + \cos x)\Big|_{\frac{5\pi}{4}}^{2\pi}$$

$$= (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{2}) + (1 + \sqrt{2})$$

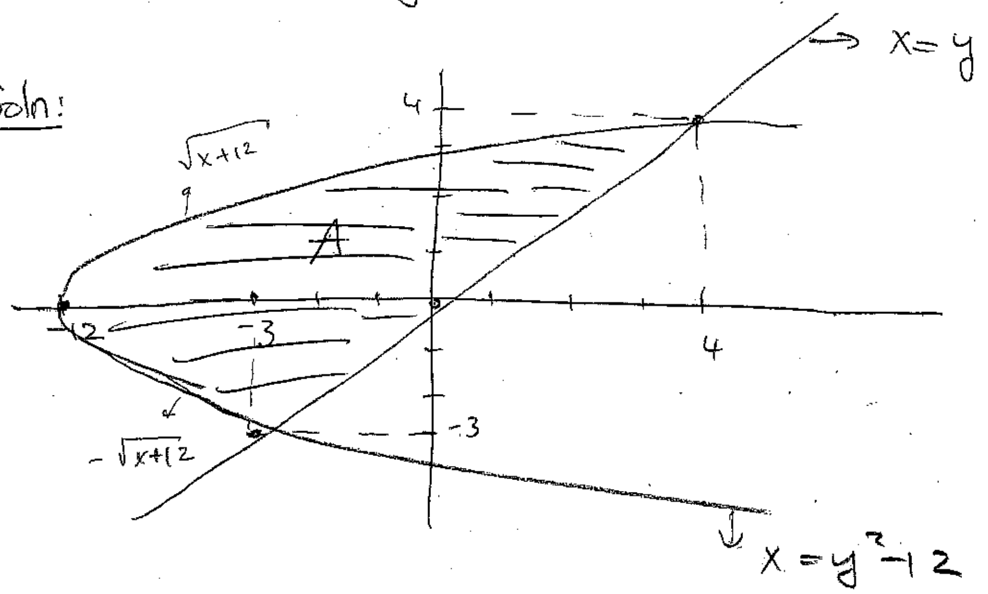
$$= 4\sqrt{2} \,//$$



$$\boxed{A = \int_c^d (g(y) - f(y))\, dy}$$

Ex: Find the area of the plane region lying to the right of the parabola $x = y^2 - 12$ and to the left of the straight line $y = x$

Soln:



To find intersection

$$y^2 - 12 = y$$

$$y^2 - y - 12 = 0$$

$$(y-4)(y+3) = 0 \qquad y = 4$$

$$y = -3$$

when $-3 < y < 4$ $\quad y^2 - 12 < y$.

$$A = \int_{-3}^{4} y - (y^2 - 12)\, dy = \left. \frac{y^2}{2} - \frac{y^3}{3} + 12y \right|_{-3}^{4} = \frac{343}{6} //$$

OR if integrate in x-direction.

$$A = \int_{-12}^{-3} \sqrt{x+12} - (-\sqrt{x+12})\, dx + \int_{-3}^{4} (\sqrt{12+x} - x)\, dx.$$

Chapter 6 Techniques of Integration:

6.1 Integration by Parts:

Suppose $u(x)$ and $v(x)$ are two differentiable functions,

By Product Rule,

$$\frac{d}{dx}\left(u(x)v(x)\right) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}.$$

Integrate

$$\int \frac{d}{dx}(u(x)v(x)) = \int u(x)\frac{dv}{dx}dx + \int v(x)\frac{du}{dx}dx.$$

$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx.$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex: $\int x e^x\,dx$

Soln: $u = x \qquad dv = e^x$

$du = dx \qquad v = e^x$

$$\int x e^x\,dx = x e^x - \int e^x\,dx = xe^x - e^x + C.$$

Ex: Use integration by parts to evaluate

a-) $\int \ln x \, dx$ c-) $\int x \tan^{-1} x \, dx$

b.) $\int x^2 \sin x \, dx$ d-) $\int \sin^{-1} x \, dx$

a.) $u = \ln x$ $dv = dx$

$du = \frac{1}{x} dx$ $v = x$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx = x \ln x - x + C \quad //.$$

b.) $u = x^2$ $dv = \sin x \, dx$

$du = 2x \, dx$ $v = -\cos x$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\int x \cos x \, dx.$$

$u = x$

$du = dx$

$dv = \cos x$

$v = \sin x$

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x \, dx\right]$$

$$= -x^2 \cos x + 2\left[x \sin x + \cos x\right]$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C \quad //.$$

c-) $u = \tan^{-1} x$ $dv = x$

$du = \frac{1}{1+x^2} dx$ $v = \frac{1}{2} x^2$

$$\int x \tan^{-1} x \, dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx.$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C. \quad //$$

d.) $u = \sin^{-1} x$ $\qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\, dx$ $\qquad v = x.$

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$u = 1 - x^2$

$du = -2x\,dx.$

$-\frac{1}{2}du = x\,dx.$

$$= x\sin^{-1} x + \frac{1}{2}\int u^{-1/2}\, du.$$

$$= x \sin^{-1} x + u^{1/2} + C$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C \quad //.$$

Note: For choosing u and dv

1-) If integrand involves.

polynomial. exponential

sine

Cosine.

some integrable function.

u = poly. $\quad dv$ = rest

2.) If integrand involves

logarithm

Inverse trigon funct

or some funct. that is not readily integrable

but whose derivative is known.

u = function.

dv = rest.

Ex: Evaluate, $I = \int \sec^3 x \, dx$

Soln: $I = \int \sec^3 x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$dv = \sec^2 x$

$v = \tan x$.

$\cancel{\int \sec^3 x \, dx} =$

$I = \int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x \, (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$.

$= \sec x \tan x - I + \ln|\sec x + \tan x|$

$2I = \sec x \tan x + \ln|\sec x + \tan x|$

$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$ //

Ex: Find $I = \int e^{ax} \cos bx \, dx$.

$u = e^{ax}$

$du = a e^{ax} dx$

$dv = \cos bx \, dx$.

$v = \frac{1}{b} \sin bx$.

$I = \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx$.

$u = e^{ax}$

$du = a e^{ax}$

$dv = \sin bx$

$v = -\frac{1}{b} \cos bx$

$$I = \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b}\left[-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}\int e^{ax} \cos b\right.$$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} I$$

$$\left(1 + \frac{a^2}{b^2}\right) I = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + C$$

$$\int e^{ax} \cos bx \, dx = \frac{b e^{ax} \sin bx + a e^{ax} \cos bx}{b^2 + a^2} + C \quad //.$$

<u>Ex:</u> $\displaystyle\int_1^e x^3 (\ln x)^2 dx$

$u = (\ln x)^2 \qquad dv = x^3 dx$

$du = \dfrac{2 \ln x \, dx}{x} \qquad v = \dfrac{x^4}{4}$.

$$\int_1^e x^3 (\ln x)^2 dx = \frac{x^4}{4} (\ln x)^2 \Big|_1^e - \frac{1}{2}\int_1^e x^3 \ln x \, dx.$$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x} dx \qquad v = \dfrac{x^4}{4}$

$$= \frac{e^4}{4} (1)^2 - 0 - \frac{1}{2}\left[\frac{x^4}{4} \ln x \Big|_1^e - \frac{1}{4}\int x^3 dx \right)$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8} \frac{x^4}{4}\Big|_1^e = \frac{e^4}{32} - \frac{1}{32} + \frac{e^4}{8}$$

$$= \frac{5}{32} e^4 - \frac{1}{32} \quad //.$$

NO

Reduction Formula:

Ex: Find $\int x^4 e^{-x}\, dx$. $\longrightarrow$ need integration by part 4 times.

Soln: Let $I_n = \int x^n e^{-x}\, dx$.

$$u = x^n \qquad dv = e^{-x}\, dx.$$
$$du = n x^{n-1} \qquad v = -e^{-x}$$

$$I_n = \int x^n e^{-x}\, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x}$$

$$I_n = -x^n e^{-x} + n I_{n-1}$$

$$I_0 = \int e^{-x}\, dx = -e^{-x} + C.$$

$$I_1 = -x e^{-x} + I_0 = -x e^{-x} - e^{-x} + C_1 = -e^{-x}(x+1) + C_1$$

$$I_2 = -x^2 e^{-x} + 2 I_1 = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C_2$$
$$= -e^{-x}(x^2 + 2x + 2) + C_2.$$

$$I_3 = -x^3 e^{-x} + 3 I_2$$
$$= -x^3 e^{-x} - 3e^{-x}(x^2 + 2x + 2) + C_3$$
$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} + C_3$$

$$I_3 = -e^{-x}(x^3 + 3x^2 + 6x + 6) + C_3$$

$$I_4 = -x^4 e^{-x} + 4 I_3 = -e^{-x}(x^4 + 12x^2 + 24x + 24) + C_4.$$

Sec 6.2 Inverse Substitution:

The Inverse Trigonometric Substitution:

1-) $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$

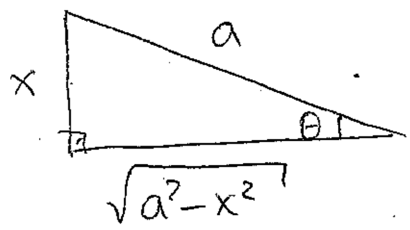
2.) $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

3.) $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\frac{x}{a}$

The Inverse sine subst:

Integrals involving $\sqrt{a^2-x^2}$ $(a>0)$ can be reduced to a simpler form by substituting

$$x = a\sin\theta \rightarrow \theta = \sin^{-1}\frac{x}{a}$$



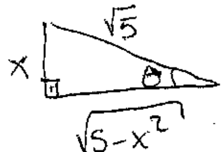
$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$$
$$= \sqrt{a^2(1-\sin^2\theta)}$$
$$= a\cos\theta.$$

Ex: Evaluate $\int \frac{1}{(5-x^2)^{3/2}}\,dx.$

$x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\, d\theta$

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx = \int \frac{\sqrt{5}\cos\theta}{(5-5\sin^2\theta)^{3/2}}\,d\theta$$



$$= \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(\cos^2\theta)^{3/2}} = \frac{1}{5}\int \sec^2\theta\, d\theta = \frac{1}{5}\tan\theta + c$$

$$= \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C =$$

NO

Ex: Evaluate $\int_0^a 2\sqrt{a^2-x^2}\,dx$.

Soln:

$$= 2\int_0^a \sqrt{a^2-x^2}\,dx$$

$x = a\sin\theta$
$dx = a\cos\theta\,d\theta$

$$= 2\int_{x=0}^{x=a} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\,d\theta$$

$\int \cos^2 x\,dx$
$= \frac{1}{2}(x + \sin x\cos x) + c$

$$= 2\int_{x=0}^{x=a} a^2\cos^2\theta\,d\theta$$

$$= 2a^2\int_{x=0}^{x=a} \cos^2\theta\,d\theta$$

$\sin\theta = \frac{x}{a}$.

$$= 2a^2\cdot\frac{1}{2}(\theta + \sin\theta\cos\theta) + c$$

$\sqrt{a^2-x^2} = \sqrt{a^2\cos^2\theta}$
$= a\cos\theta$
$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$$= a^2\left(\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{a^2}\right)\Bigg|_{x=0}^{x=a} + c$$

$$= a^2\left[\frac{\pi}{2}\right].$$

The Inverse tangent Subst:

Integrals involving $\sqrt{a^2+x^2}$ or $\frac{1}{x^2+a^2}$ $(a>0)$

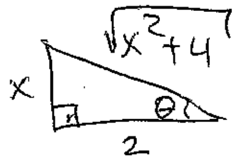
are simplified by the subst.

$\boxed{x = a\tan\theta}$ or $\boxed{\theta = \tan^{-1}\frac{x}{a}}$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sec\theta.$$

Ex: Evaluate

a.) $\int \frac{1}{\sqrt{4+x^2}}\,dx$



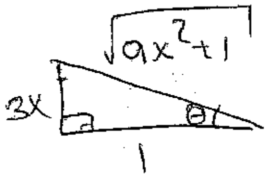
Soln: $\int \frac{1}{\sqrt{4+x^2}}\,dx$ $\qquad x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4+4\tan^2\theta}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C.$$



(b.) $\int \frac{1}{(1+9x^2)^2}\,dx$

$3x = \tan\theta$

$3\,dx = \sec^2\theta\, d\theta$

$$= \int \frac{\frac{1}{3}\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$$

$$= \frac{1}{3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{1}{3}\int \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{3}\cdot\frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C.$$

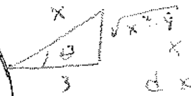
$$= \frac{1}{6}\left[\tan^{-1}3x + \frac{3x}{\sqrt{1+9x^2}}\cdot\frac{1}{\sqrt{1+9x^2}}\right] + C.$$

$$= \frac{1}{6}\tan^{-1}3x + \frac{1}{2}\,\frac{x}{1+9x^2} + C \quad //.$$

The Inverse Secant Substitution:

Integrals involving $\sqrt{x^2-a^2}$ $(a>0)$ can be simplified by using the subst.

Ex: $\int \frac{\sqrt{x^2-9}}{x}\, dx$



$x = 3\sec\theta$

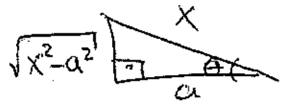
$dx = 3\sec\theta\tan\theta\, d\theta$

$\sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta-1}$
$= 3\sqrt{\tan^2\theta}$
$= 3\tan\theta$

$$\boxed{x = a\sec\theta \quad \Rightarrow \quad \theta = \sec^{-1}\frac{x}{a}}$$

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = a\sqrt{\tan^2\theta} = a|\tan\theta| = \begin{cases} a\tan\theta & x \\ -a\tan\theta & x\leq \end{cases}$$

Ex: Find $I = \int \frac{dx}{\sqrt{x^2-a^2}}$ $(a>0)$.



$$\int \frac{dx}{\sqrt{x^2-a^2}}$$

$x = a\sec\theta$ $\quad dx = a\sec\theta\tan\theta\, d\theta$

i) $x\geq a$ $\quad \sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a\tan\theta$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| \underbrace{+ \ln a + C}_{C_1}$$

if $x \leq -a$, $\quad u = -x \Rightarrow u \geq a$

$du = -dx$.

$$\int \frac{dx}{\sqrt{x^2-a^2}} = -\int \frac{du}{\sqrt{u^2-a^2}} = -\ln\left|u + \sqrt{u^2-a^2}\right| + C_1$$

$$= \ln \left| \frac{1}{-x+\sqrt{x^2-a^2}} \cdot \frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}} \right| + c_1$$

$$= \ln \left| \frac{x+\sqrt{x^2-a^2}}{-a^2} \right| + c_1 = \ln \left| x+\sqrt{x^2-a^2} \right| + C_2.$$

$$C_2 = c_1 - 2\ln a.$$

Completing the Square:

$Ax^2 + Bx + C \rightarrow$ found in integrands.

$$\underset{=2a}{} \quad a = \frac{B}{2A} \qquad (x+a)^2 = x^2 + 2ax + a^2$$

$$Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right)$$

$$= A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} + \frac{C}{A} - \frac{B^2}{4A^2}\right)$$

$$= A\left(x + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A}.$$

then substitute $u = x + \dfrac{B}{2A}$

Ex: Evaluate,

a.) $\displaystyle\int \frac{1}{\sqrt{2x - x^2}}\, dx$

$u = x - 1$
$du = dx.$

$$= \int \frac{1}{\sqrt{1-(1-2x+x^2)}}\, dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\, dx = \int \frac{1}{\sqrt{1-u^2}}\, du.$$

$$= \sin^{-1} u + c = \sin^{-1}(x-1) + c \quad //.$$

b.) $\displaystyle\int \frac{x}{4x^2+12x+13}\,dx$

$$= \int \frac{x\,dx}{4(x^2+3x+\frac{9}{4}+1)} = \int \frac{x\,dx}{4\left[(x+\frac{3}{2})^2+1\right]}$$

$u = x+\frac{3}{2}$

$du = dx$

$x = u-\frac{3}{2}$

$$= \frac{1}{4}\int \frac{u\,du}{u^2+1} - \frac{3}{8}\int \frac{du}{u^2+1}$$

$v = u^2+1$

$dv = 2u\,du$

$\frac{1}{2}dv = u\,du$

$$= \frac{1}{8}\int \frac{dv}{v} - \frac{3}{8}\tan^{-1}u.$$

$$= \frac{1}{8}\ln|v| - \frac{3}{8}\tan^{-1}u$$

$$= \frac{1}{8}\ln\left((x+\tfrac{3}{2})^2+1\right) - \frac{3}{8}\tan^{-1}(x+\tfrac{3}{2})$$

Other Inverse Substitutions.

Integrals involving $\sqrt{ax+b}$. $\rightarrow$ will simpler

by substituting $u^2 = ax+b$.

Ex: $\displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx =$

$u = \sqrt{2x}$

$u^2 = 2x$.

$2u\,du = 2\,dx$

$u\,du = dx$

$$= \int \frac{u\,du}{1+u} = \int \frac{1+u-1}{1+u}\,du$$

$$= \int \left(1 - \frac{1}{1+u}\right)du = \int du - \int \frac{1}{1+u}\,du$$

$v = 1+u$

$dv = du$

$$= u - \int \frac{dv}{v} = u - \ln|v| + C.$$

$$= \sqrt{2x} - \ln(1+\sqrt{2x}) + C$$

Note: Integrals involving $\sqrt[n]{ax+b}$.

substitute $ax+b = u^n$

$a\,dx = nu^{n-1}du$.

Ex: $$\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx =$$

$x = -\frac{1}{3} \Rightarrow u = 1$

$x = 2 \Rightarrow u = 2$.

$u^3 = 3x+2$.

$\cancel{3}u^2du = \cancel{3}dx$.

$u^2du = dx$

$x = \frac{u^3-2}{3}$

$$= \int_1^2 \frac{u^3-2}{3\,u}\,u^2du$$

$\sqrt[3]{3x+2}$

$$= \frac{1}{3}\int_1^2 (u^4 - 2u)\,du$$

$$= \frac{1}{3}\left[\frac{u^5}{5} - \frac{2u^2}{2}\Big|_1^2\right] = \frac{16}{15}.$$

1, Note: If more than one fractional power is present, we could eliminate all of Them at once.

Ex: Evaluate $$\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$$

$2|3|^2_3$, 6 LCM of $2,3$.

$x = u^6$

$dx = 6u^5du$

$$= 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2}{1+u^2}\,du$$

$$= 6\int \left(1 - \frac{1}{1+u^2}\right)du = 6u - \tan^{-1}u + C.$$

$$= 6x^{1/6} - 6\tan^{-1}x^{1/6} + C.$$

The $\tan\left(\frac{\theta}{2}\right)$ Subst:

With this substitution we can transform an integral whose integrand is a rational function of $\sin\theta$ and $\cos\theta$ into a rational function of x.

if $\boxed{x = \tan\frac{\theta}{2}}$ $\Rightarrow$

$$\boxed{\begin{aligned} \cos\theta &= \frac{1-x^2}{1+x^2} \\ \sin\theta &= \frac{2x}{1+x^2} \\ d\theta &= \frac{2\,dx}{1+x^2} \end{aligned}}$$

Ex: $\displaystyle\int \frac{1}{2+\cos\theta}\,d\theta$

$\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$x = \tan\frac{\theta}{2}$

$\cos\theta = \dfrac{1-x^2}{1+x^2}$

$d\theta = \dfrac{2\,dx}{1+x^2}$

$$= \int \frac{\frac{2\,dx}{1+x^2}}{2+\frac{1-x^2}{1+x^2}} = 2\int \frac{1}{3+x^2}\,dx$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + c.$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + c \quad //.$$

Joint Universities Preliminary Examinations Board (JUPEB)

2014 EXAMINATIONS RESULTS

Exam Number: 14206716YC
Last Name: OGUNGBURE
Other Name(s): GODWIN L
Institution: FUTA

Subject	Code	Grade
Chemistry	**CHE**	**C**
Mathematics	**MAT**	**B**
Physics	**PHY**	**B**

Prof. Duro Ajeyalemi
Interim Coordinator
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18/02/2015
Date