

Sec 6.3 Integrals of Rational Functions

$$\int \frac{P(x)}{Q(x)} \quad \text{where } P(x), Q(x) \text{ are polynomials}$$

where $\deg P < \deg Q$.

if $\deg P \geq \deg Q \Rightarrow$ use long division

Ex: Evaluate $\int \frac{x^3 + 3x^2}{x^2 + 1} dx$

Soln: $\deg$ of $P(x) = 3$
 $\deg$ of $Q(x) = 2$.

$$\begin{array}{r} x^2 + 1 \overline{) x^3 + 3x^2} \\ \underline{-x^3 + x} \\ 2x^2 - x \\ \underline{-2x^2 + 3} \\ -x - 3 \end{array}$$

$$\frac{x^3 + 3x^2}{x^2 + 1} = x + 3 + \frac{-x - 3}{x^2 + 1}$$

$$\int \frac{x^3 + 3x^2}{x^2 + 1} dx = \int (x + 3) dx + \int \frac{-x}{x^2 + 1} dx - 3 \int \frac{1}{x^2 + 1} dx$$

$$= \frac{x^2}{2} + 3x - \frac{1}{2} \ln(x^2 + 1) - 3 \tan^{-1} x + C$$

$$\begin{aligned} u &= x^2 + 1 \\ du &= 2x dx \\ \frac{1}{2} du &= x dx \end{aligned}$$

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{du}{u} = \ln u$$

Ex: Evaluate $\int \frac{x}{2x-1} dx$.

$$\deg P(x) = \deg Q(x).$$

$$\begin{array}{r} \frac{1}{2} \\ 2x-1 \overline{) x} \\ \underline{-x + \frac{1}{2}} \\ \frac{1}{2} \end{array}$$

$$\begin{aligned} \frac{x}{2x-1} &= \frac{1}{2} + \frac{\frac{1}{2}}{2x-1} \\ &= \frac{1}{2} \left(1 + \frac{1}{2x-1} \right) \end{aligned}$$

$$\int \frac{x}{2x-1} dx = \frac{1}{2} \left[\int 1 dx + \int \frac{1}{2x-1} dx \right]$$

$$= \frac{1}{2} \left[x + \frac{1}{2} \ln |2x-1| \right] + C$$

$$u = 2x-1$$

$$du = 2dx$$

$$\frac{1}{2} du = dx$$

$$= \frac{1}{2} x + \frac{1}{4} \ln |2x-1| + C.$$

$$\begin{aligned} \int \frac{1}{2x-1} dx &= \frac{1}{2} \int \frac{1}{u} du \\ &= \ln u \end{aligned}$$

what about $\int \frac{P(x)}{Q(x)} dx$ when $\deg P(x) < \deg Q(x)$?

case i-) Linear Denominator ($\deg Q(x) = 1$).

$$\begin{aligned} Q(x) &= ax+b \\ P(x) &= c \end{aligned}$$

$$\int \frac{c}{ax+b} dx = \frac{c}{a} \int \frac{1}{u} du$$

$$u = ax+b$$

$$du = a dx$$

$$\frac{1}{a} du = dx$$

$$= \frac{c}{a} \ln |ax+b| + C$$

if $c = 1$.

$$\boxed{\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C.}$$

Thm: Partial fraction decomposition of rational functions

Let P and Q are real polynomials, real coefficients

$$\deg P < \deg Q$$

$$a.) \quad Q(x) = K(x-a_1)^{m_1}(x-a_2)^{m_2} \dots (x-a_j)^{m_j} (x^2+b_1x+c_1)^{n_1} \dots (x^2+b_kx+c_k)^{n_k}$$

$$\deg Q = m_1 + m_2 + \dots + m_j + 2n_1 + 2n_2 + \dots + 2n_k$$

$$b.) \quad \frac{P(x)}{Q(x)} = \text{sum of partial fractions.}$$

i-) for factors $(x-a)^m$ of $Q(x)$

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_m}{(x-a)^m}$$

ii-) for factors $(x^2+bx+c)^n$ of $Q(x)$

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \dots + \frac{B_nx+C_n}{(x^2+bx+c)^n}$$

case(i-) Quadratic Denominator ($\deg Q(x)=2$).

$$Q(x) \begin{cases} \nearrow x^2+a^2 \\ \searrow x^2-a^2 \end{cases} \quad \text{then } P(x)=ax+b \text{ at most.}$$

Four cases.

$$1.) \int \frac{x dx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2) + C.$$

$$\boxed{u = x^2+a^2}$$

$$du = 2x dx.$$

$$\frac{1}{2} du = x dx.$$

$$2.) \int \frac{x dx}{x^2-a^2} = \frac{1}{2} \ln |x^2-a^2| + C.$$

substit $u = x^2-a^2$.

$$3.) \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

$$4.) \int \frac{dx}{x^2-a^2}.$$

write denominator as sum of two fractions with linear denominator

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{Ax+aA+Bx-Ba}{x^2-a^2}.$$

$$\frac{1}{x^2-a^2} = \frac{(A+B)x + aA - Ba}{x^2-a^2}$$

$$A+B=0 \Rightarrow A=-B.$$

$$aA - Ba = 1$$

$$-2Ba = 1$$

$$\boxed{B = -\frac{1}{2a}}$$

$$\boxed{A = \frac{1}{2a}}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \frac{1}{x-a} dx - \frac{1}{2a} \int \frac{1}{x+a} dx.$$

$$= \frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C$$

$$\boxed{\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C.}$$

Partial Fractions:

This technique, writing complicated fraction as a sum of simpler fractions is called method of partial fractions

$$Q(x) = (x-a_1)(x-a_2) \dots (x-a_n) \rightarrow n \text{ distinct linear factors.}$$

$$\deg Q(x) = n$$

If $\deg P(x) < n \Rightarrow \frac{P(x)}{Q(x)}$ has a partial fraction decomposition

as:

$$\boxed{\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}.}$$

Ex: Evaluate $\int \frac{x+4}{x^2-5x+6} dx$

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{Ax-3A+Bx-2B}{(x-2)(x-3)}$$

$$\frac{x+4}{(x-2)(x-3)} = \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}$$

$$A+B=1$$

$$-3A-2B=4$$

$$A=1-B$$

$$-3(1-B)-2B=4$$

$$-3+3B-2B=4$$

$$B=7 //$$

$$A=-6 //$$

Ex: Evaluate

$$\int \frac{4x^2+13x-9}{x^3+2x^2-3} dx$$

Ex: Evaluate

$$\int \frac{x^2-x-21}{2x^3-x^2+8x-4} dx$$

$$\frac{x^2(2x-1)+4(2x-1)}{(2x-1)(x^2+4)}$$

$$= \frac{A}{2x-1} + \frac{Bx+C}{x^2+4}$$

$$\frac{x+4}{(x-2)(x-3)} = -\frac{6}{x-2} + \frac{7}{x-3}$$

$$\int \frac{x+4}{(x-2)(x-3)} dx = -6 \int \frac{1}{x-2} dx + 7 \int \frac{1}{x-3} dx$$

$$= -6 \ln|x-2| + 7 \ln|x-3| + C //$$

Ex: Evaluate $I = \int \frac{x^3+2}{x^3-x} dx$

Soln:

$$\frac{x^3+2}{x^3-x} = \frac{1}{x^3-x} \cdot \frac{x^3+2}{-x^3+x} = \frac{x+2}{x+2}$$

$$\frac{x^3+2}{x^3-x} = 1 + \frac{x+2}{x^3-x}$$

$$\int \frac{x^3+2}{x^3-x} dx = \int \left(1 + \frac{x+2}{x^3-x}\right) dx = x + \int \frac{x+2}{x^3-x}$$

$$\frac{x+2}{x^3-x} = \frac{x+2}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{Ax^2-A+Bx^2+Bx+Cx^2-Cx}{x(x-1)(x+1)}$$

$$\frac{x+2}{x(x-1)(x+1)} = \frac{(A+B+C)x^2+(B-C)x-A}{x(x-1)(x+1)}$$

$$A+B+C=0$$

$$B-C=1$$

$$-A=2$$

$$B+C=2$$

$$B-C=1$$

$$\Rightarrow \boxed{A=-2}$$

$$2B=3$$

$$\boxed{B=\frac{3}{2}}$$

$$\boxed{C=\frac{1}{2}}$$

$$\frac{x+2}{x^3-x} = -2 \frac{1}{x} + \frac{3}{2} \frac{1}{x-1} + \frac{1}{2} \frac{1}{x+1}$$

$$I = x - 2 \int \frac{1}{x} dx + \frac{3}{2} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x+1} dx$$

$$= x - 2 \ln|x| + \frac{3}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C //$$

Ex: Evaluate $\int \frac{2+3x+x^2}{x(x^2+1)} dx$.

Soln:

$$\deg P(x)=2$$

$$\deg Q(x)=3.$$

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

$$\boxed{A=2}$$

$$\boxed{C=3}$$

$$A+B=1 \Rightarrow \boxed{B=-1}$$

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{2}{x} + \frac{-x+3}{x^2+1}$$

$$\int \frac{2+3x+x^2}{x(x^2+1)} dx = 2 \int \frac{1}{x} dx + 3 \int \frac{1}{x^2+1} dx - \int \frac{x}{x^2+1} dx$$

$$= 2 \ln|x| + 3 \tan^{-1}x - \frac{1}{2} \ln(x^2+1) + C$$

Ex! Evaluate $I = \int \frac{1}{x^3+1} dx$.

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{Ax^2 - Ax + A + Bx^2 + Cx + C}{(x+1)(x^2-x+1)}$$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{(A+B)x^2 + (B-A+C)x + A+C}{(x+1)(x^2-x+1)}$$

$$A+B=0 \Rightarrow A=-B$$

$$B-A+C=0 \quad -2A+C=0$$

$$A+C=1 \quad -A-C=-1$$

$$-3A=-1$$

$$A = -\frac{1}{3}$$

$$B = \frac{1}{3}$$

$$C = \frac{2}{3}$$

$$\int \frac{1}{x^2+1} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\frac{1}{(x+1)(x^2-x+1)} = -\frac{1}{3} \frac{1}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$$

$$\int \frac{1}{(x+1)(x^2-x+1)} dx = \frac{1}{3} \int \frac{1}{x+1} dx - \frac{1}{3} \int \frac{x-2}{x^2-x+1} dx$$

$$= \frac{1}{3} \ln|x+1| - \frac{1}{3} \int \frac{x - \frac{1}{2} - \frac{3}{2}}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx$$

$$x - \frac{1}{2} = u$$

$$dx = du$$

$$u^2 + \frac{3}{4} = u$$

$$2u du = du$$

$$u du = \frac{du}{2}$$

$$\int \frac{u - \frac{3}{2}}{u^2 + \frac{3}{4}} du$$

$$= \int \frac{u}{u^2 + \frac{3}{4}} du$$

$$- \frac{3}{2} \int \frac{du}{u^2 + \frac{3}{4}}$$

$$= \frac{1}{2} \int \frac{du}{u} - \frac{3}{2} \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) = \frac{1}{2} \ln|u| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)$$

Ex: Evaluate $\int \frac{1}{x(x-1)^2} dx$.

→ linear factors of Q are repeated.

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\frac{1}{x(x-1)^2} = \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2} = \frac{(A+B)x^2 + (-2A-B+C)x + A}{x(x-1)^2}$$

$$A+B=0$$

$$-2A-B+C=0$$

$$\boxed{A=1}$$

$$\Rightarrow \boxed{B=-1} \quad \boxed{C=1}$$

$$u = x-1$$

$$(x-1)^{-2} = u^{-2} = \frac{u^{-1}}{-1}$$

$$\int \frac{1}{x(x-1)^2} dx = \int \frac{dx}{x} - \int \frac{1}{x-1} dx + \int \frac{1}{(x-1)^2} dx$$

$$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$

$$= \ln \left| \frac{x}{x-1} \right| - \frac{1}{x-1} + C$$

Ex: Evaluate $I = \int \frac{x^2+2}{4x^5+4x^3+x} dx$

→ quadratic factors of Q are repeated

$$\frac{x^2+2}{4x^5+4x^3+x} = \frac{x^2+2}{x(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$

$$= \frac{A(4x^4+4x^2+1) + (Bx+C)(2x^3+x) + (Dx+E)x}{x(2x^2+1)^2}$$

$$\frac{x^2+2}{x(2x^2+1)^2} = \frac{A(4x^4+4x^2+1) + B(2x^4+x^2) + C(2x^3+x) + Dx^2 + Ex}{x(2x^2+1)^2}$$

$$4A + 2B = 0$$

$$2C = 0$$

$$4A + B + D = 1$$

$$C + E = 0$$

$$A = 2$$

$$A = 2$$

$$B = -4$$

$$C = 0$$

$$D = -3$$

$$E = 0$$

$$\int \frac{x^2+2}{4x^5+4x^3+x} dx = 2 \int \frac{1}{x} dx - 4 \int \frac{x dx}{2x^2+1} - 3 \int \frac{x dx}{(2x^2+1)^2}$$

$$= 2 \ln|x| - \frac{1}{4} \int \frac{du}{u} - \frac{3}{4} \int \frac{du}{u^2}$$

$$u = 2x^2+1$$

$$du = 4x dx$$

$$\frac{1}{4} du = x dx$$

$$= 2 \ln|x| - \ln|u| + \frac{3}{4u} + C$$

$$= 2 \ln|x| - \ln(2x^2+1) + \frac{3}{4} \frac{1}{2x^2+1} + C$$

$$= \ln \left(\frac{x^2}{2x^2+1} \right) + \frac{3}{4(2x^2+1)} + C //$$

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Sec 6.5 Improper Integrals:

①

Defn: Improper integrals of type I

If f is continuous on $[a, \infty)$ then

$$\int_a^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_a^R f(x) dx$$

if f is continuous on $(-\infty, b]$ then

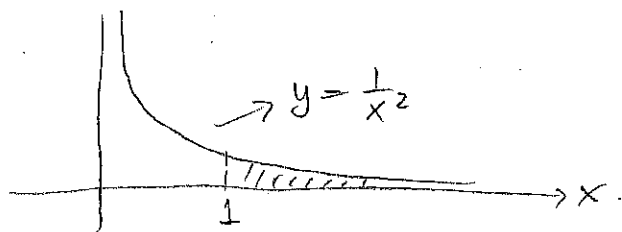
$$\int_{-\infty}^b f(x) dx = \lim_{R \rightarrow -\infty} \int_R^b f(x) dx.$$

In either cases, if the limit exists, $\Rightarrow$ the improper integral converges.

if limit does not exist $\Rightarrow$ improper integral diverges.

if limit is ∞ or $-\infty \Rightarrow$ improper integral diverges to infinity
(or diverges negative infinity)

Ex: Find the area of A the region lying under the curve $y = \frac{1}{x^2}$ and above the x -axis to the right of $x = 1$.



$$A = \int_1^{\infty} \frac{dx}{x^2} = \lim_{R \rightarrow \infty} \int_1^R \frac{dx}{x^2} = \lim_{R \rightarrow \infty} \left(-\frac{1}{x} \right) \Big|_1^R$$

$$= \lim_{R \rightarrow \infty} \left(-\frac{1}{R} + 1 \right) = 1 \quad // \quad \text{since limit exists.}$$

improper integral converges.

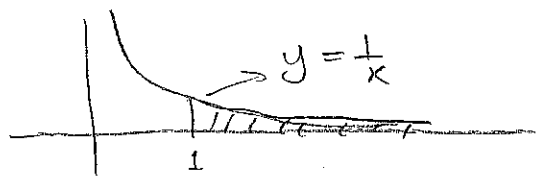
Ex: $\int_0^{\infty} e^{-2x} dx$

$$= \lim_{R \rightarrow \infty} \int_0^R e^{-2x} dx = \lim_{R \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_0^R$$

$$= \lim_{R \rightarrow \infty} \left[-\frac{1}{2e^{2R}} + \frac{1}{2} \frac{1}{e^0} \right] = \frac{1}{2}$$

$\downarrow$ $\downarrow$
 0 1

Ex: Find the area of the region under $y = \frac{1}{x}$ above $y = 0$ and to the right of $x = 1$.



$$\int_1^{\infty} \frac{1}{x} dx = \lim_{R \rightarrow \infty} \int_1^R \frac{dx}{x} = \lim_{R \rightarrow \infty} \ln x \Big|_1^R = \lim_{R \rightarrow \infty} \ln R - \ln 1 = \infty$$

improper
integral diverges
to infinity

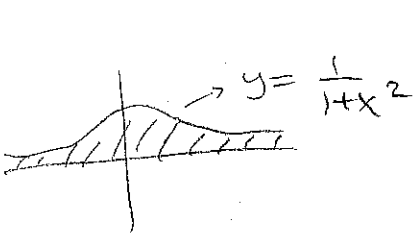
Note: $\int_{-\infty}^{\infty} f(x) dx$, f is continuous on the real line, (2)

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx.$$

Ex: $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

$\frac{1}{1+x^2} \rightarrow$ even function.

Soln: $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$



$$= 2 \int_0^{\infty} \frac{1}{1+x^2} dx.$$

$$= 2 \lim_{R \rightarrow \infty} \int_0^R \frac{1}{1+x^2} dx = 2 \lim_{R \rightarrow \infty} [\tan^{-1} R - \tan^{-1} 0]$$

$$= 2 \lim_{R \rightarrow \infty} [\tan^{-1} R - 0] = 2 \tan^{-1} \infty$$

$$= 2 \cdot \frac{\pi}{2} = \pi //$$

Ex: $\int_0^{\infty} \cos x dx = \lim_{R \rightarrow \infty} \int_0^R \cos x dx = \lim_{R \rightarrow \infty} (\sin R - \sin 0)$

$$= \lim_{R \rightarrow \infty} \sin R$$

limit does not exist
improper integral diverges.

Defn: Improper Integrals of Type II

If f is continuous on the interval $(a, b]$ and is unbounded near a ,

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

If f is continuous on $[a, b)$ and unbounded near b

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

Those integrals may converge, diverge, diverge to infinity or diverge to negative infinity.

Ex: Find area of the region S lying under $y = \frac{1}{\sqrt{x}}$ above the x -axis between $x=0$ and $x=1$

Soln:

$$A = \int_0^1 \frac{1}{\sqrt{x}} dx \quad \frac{1}{\sqrt{x}} \text{ unbounded near } x=0.$$

$$A = \lim_{c \rightarrow 0^+} \int_c^1 x^{-1/2} dx = \lim_{c \rightarrow 0^+} \left(2x^{1/2} \right) \Big|_c^1 = \lim_{c \rightarrow 0^+} (2 - 2\sqrt{c}) = 2$$

(3)

Note: It may be necessary to break an improper integral into several improper integrals if it is improper at both endpoints or at points inside the interval of integration.

Ex: $\int_{-1}^1 \frac{\ln|x| dx}{\sqrt{1-x^2}} = \int_{-1}^0 \frac{\ln|x| dx}{\sqrt{1-x^2}} + \int_0^{1/2} \frac{\ln|x| dx}{\sqrt{1-x^2}} + \int_{1/2}^1 \frac{\ln|x| dx}{\sqrt{1-x^2}}$

Ex1 $\int_0^1 \frac{1}{x} dx$

$= \lim_{c \rightarrow 0^+} \int_c^1 \frac{1}{x} dx = \lim_{c \rightarrow 0^+} (\ln x \Big|_c^1) = \lim_{c \rightarrow 0^+} (\ln 1 - \ln c) = \infty$

diverges to infinity.

Ex2 $\int_0^2 \frac{1}{\sqrt{2x-x^2}} dx$

$-x^2 + 2x = -(x^2 - 2x) = -(x-1)^2 - 1 = 1 - (x-1)^2$

$\int_0^2 \frac{1}{\sqrt{1-(x-1)^2}} dx = \int_{-1}^1 \frac{1}{\sqrt{1-u^2}} du$ ~~unbounded at x=1~~
 $\frac{1}{1-u^2}$ symmetric.

$u = x-1 \quad x=2 \quad u=1$
 $du = dx \quad x=0 \quad u=-1$

$= 2 \int_0^1 \frac{1}{\sqrt{1-u^2}} du$

$= 2 \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{\sqrt{1-u^2}} du$

$= 2 \lim_{c \rightarrow 1^-} (\sin^{-1} u \Big|_0^c) = 2 \lim_{c \rightarrow 1^-} \sin^{-1} c = 2 \cdot \frac{\pi}{2} = \pi$

converges to π

Ex 3 $\int_0^1 \ln x \, dx =$

$$= \lim_{c \rightarrow 0^+} (x \ln x - x) \Big|_c^1 = \lim_{c \rightarrow 0^+} (1 \cdot 0 - 1 - c \ln c + c)$$

$$= -1 - \lim_{c \rightarrow 0^+} \frac{\ln c}{\frac{1}{c}} \quad \left[\frac{-\infty}{\infty} \right].$$

$$= -1 - \lim_{c \rightarrow 0^+} \frac{\frac{1}{c}}{-\left(\frac{1}{c^2}\right)} = -1 - \lim_{c \rightarrow 0^+} (-c) = -1 + 0 = -1 //$$

Thm: P-integrals.

If $0 < a < \infty$

$$a.) \int_a^{\infty} x^{-p} \, dx = \begin{cases} \text{converges to } \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ \text{diverges to } \infty & \text{if } p \leq 1 \end{cases}$$

$$b.) \int_0^a x^{-p} \, dx = \begin{cases} \text{converges to } \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \text{diverges to } \infty & \text{if } p \geq 1 \end{cases}$$

Ex: $\int_0^2 f(x) \, dx$ where $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ x-1 & \text{if } 1 < x \leq 2. \end{cases}$

$$\int_0^2 f(x) \, dx = \int_0^1 \frac{1}{\sqrt{x}} \, dx + \int_1^2 (x-1) \, dx$$

$$= \lim_{c \rightarrow 0^+} \int_c^1 x^{-1/2} \, dx + \left(\frac{x^2}{2} - x \right) \Big|_1^2 = \lim_{c \rightarrow 0^+} 2\sqrt{x} \Big|_c^1 + 0 - \frac{1}{2} + 1$$

$$= \lim_{c \rightarrow 0^+} (2 - 2\sqrt{c}) + \frac{1}{2} = \frac{5}{2} //$$

Estimating Convergence and Divergence:

if integral can't be evaluated because antiderivative can't be found. (4)

Thm: Comparison thm for integrals:

Then we can say whether they converge or diverge.

Let $-\infty \leq a < b \leq \infty$ and suppose f and g are continuous on (a, b) and $0 \leq f(x) \leq g(x)$.

then if $\int_a^b g(x) dx$ converges $\Rightarrow \int_a^b f(x) dx$ also converges.

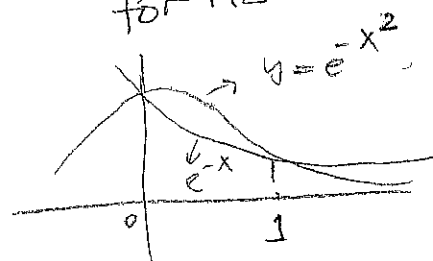
$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

if $\int_a^b f(x) dx$ diverges to $\infty \Rightarrow$ so does $\int_a^b g(x) dx$.

Ex: Show that $\int_0^{\infty} e^{-x^2} dx$ converges and find an upper-bound for its value.

Soln:

$$e^{-x^2} \leq e^{-x} \quad \text{for } x \geq 1.$$



$$0 \leq x \leq 1 \quad 0 < e^{-x^2} \leq 1.$$

$$\int_0^1 dx < \int_0^1 e^{-x^2} dx \leq \int_0^1 1 dx$$

$$0 < \int_0^1 e^{-x^2} dx \leq 1$$

$$0 \leq \int_1^{\infty} e^{-x^2} dx \leq \int_1^{\infty} e^{-x} dx = \lim_{R \rightarrow \infty} \left(\frac{e^{-x}}{-1} \right) \Big|_1^R = \lim_{R \rightarrow \infty} \left(-\frac{1}{e^R} + \frac{1}{e} \right) = \frac{1}{e}.$$

$$\int_0^{\infty} e^{-x^2} dx \text{ converges } \leq 1 + \frac{1}{e}$$

Ex: Determine whether $\int_0^{\infty} \frac{dx}{\sqrt{x+x^3}}$ converges.

Soln:

$$\int_0^{\infty} \frac{dx}{\sqrt{x+x^3}} = \underbrace{\int_0^1 \frac{dx}{\sqrt{x+x^3}}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}}_{I_2}$$

$$0 < x \leq 1 \quad \sqrt{x+x^3} > \sqrt{x}$$

$$\int_0^1 \frac{dx}{\sqrt{x+x^3}} < \int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{-1/2} dx = \frac{1}{1-\frac{1}{2}} = 2 \quad \text{by thm 2}$$

$$1 \leq x < \infty \quad \sqrt{x+x^3} > \sqrt{x^3}$$

$$\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}} < \int_1^{\infty} \frac{1}{\sqrt{x^3}} dx = \int_1^{\infty} x^{-3/2} dx = \frac{1}{\frac{3}{2}-1} = \frac{1}{\frac{1}{2}} = 2$$

(by thm 2)

$$\int_0^{\infty} \frac{dx}{\sqrt{x+x^3}} < 4$$

↓
converges

Ex: Determine whether $\int_2^{\infty} \frac{dx}{\ln x}$ converges.

$$\ln x < x \quad \text{for } x \geq 2$$

$$\int_2^{\infty} \frac{dx}{\ln x} > \int_2^{\infty} \frac{1}{x} dx = \int_2^{\infty} x^{-1} dx = \text{diverges by thm 2}$$

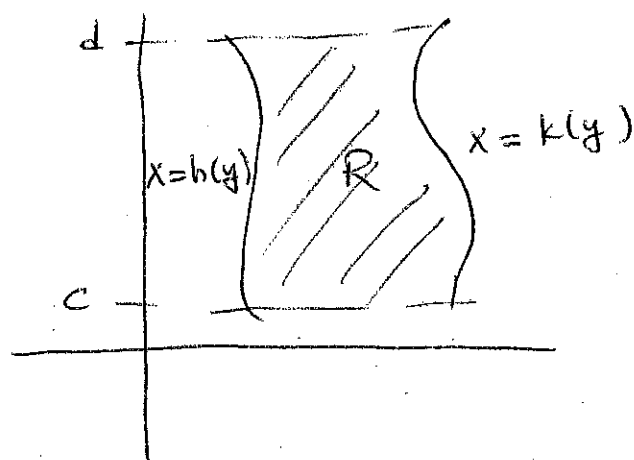
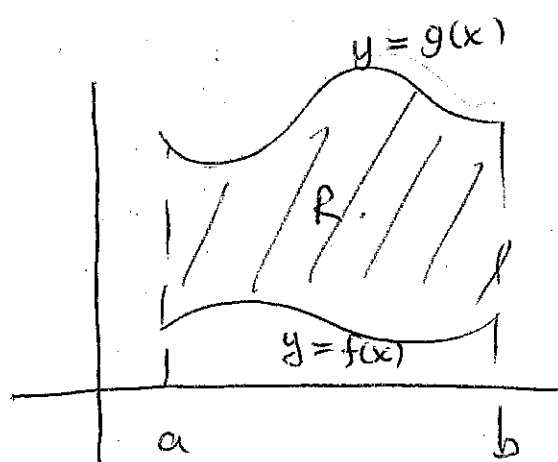
So $\int_2^{\infty} \frac{dx}{\ln x}$ also diverges by thm 2

Chapter 7 Applications of Integration:

Sec 7.1 Volumes of Solids of Revolution.

Defn: The volume V of a solid between $x=a$ and $x=b$ having cross-sectional area $A(x)$ at position x is

$$V = \int_a^b A(x) dx$$



Method 1 Plane slices.

If R is rotated about x -axis

$$V = \pi \int_a^b ((g(x))^2 - (f(x))^2) dx$$

If R is rotated about y -axis

$$V = \pi \int_c^d ((k(y))^2 - (h(y))^2) dy$$

Method 2 Using Cylindrical Shells

i.) if R is rotated about x -axis

$$V = 2\pi \int_a^d y(k(y) - h(y)) dy$$

ii.) if R is rotated about y -axis

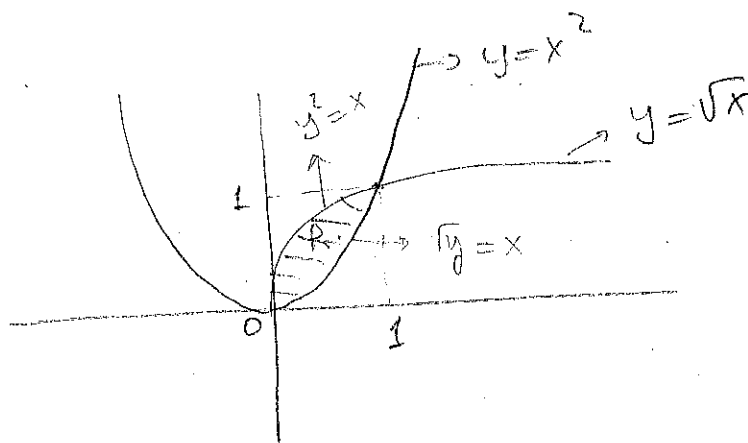
$$V = 2\pi \int_a^b x(g(x) - f(x)) dx$$

Ex: Find the volume of the solid generated by rotating the region bounded by $y = x^2$ and $y = \sqrt{x}$ between $x=0$ and $x=1$ by using plane slices method, by

a.) rotating about x -axis

b.) " " " y -axis.

Soln:



$$\begin{aligned} x^2 &= \sqrt{x} \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \\ x(x-1)(x^2 + x + 1) &= 0 \\ x=0 \quad x=1 \end{aligned}$$

ii)

a.) rotating about x-axis

$$\begin{aligned}
 V &= \pi \int_a^b (g(x))^2 - (f(x))^2 dx \\
 &= \pi \int_0^1 ((\sqrt{x})^2 - (x^2)^2) dx \\
 &= \pi \int_0^1 (x - x^4) dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \\
 &= \pi \left[\frac{1}{2} - \frac{1}{5} \right] = \frac{3\pi}{10} //
 \end{aligned}$$

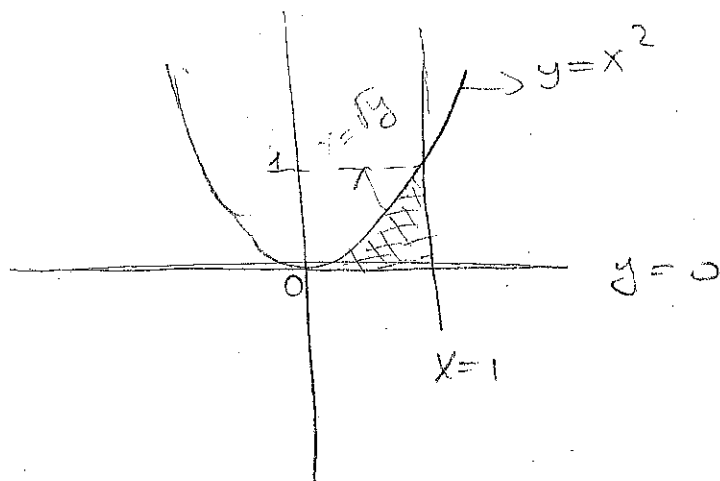
b.) rotating about y-axis

$$\begin{aligned}
 V &= \pi \int_c^d ((k(y))^2 - (h(y))^2) dy \\
 V &= \pi \int_0^1 ((\sqrt{y})^2 - (y^2)^2) dy \\
 V &= \pi \int_0^1 (y - y^4) dy = \pi \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1 = \frac{3\pi}{10} //
 \end{aligned}$$

Ex: Find the volume of the solid generated by rotating the region bounded by $y=x^2$, $y=0$ and $x=1$ by using cylindrical shells method.

a.) rotate around x-axis

b.) " " y-axis



a-) around x-axis.

$$V = 2\pi \int_0^d y (k(y) - h(y)) dy.$$

$$= 2\pi \int_0^1 y (1 - \sqrt{y}) dy.$$

$$= 2\pi \int_0^1 (y - y^{3/2}) dy = 2\pi \left[\frac{y^2}{2} - \frac{2y^{5/2}}{5} \right]_0^1$$

$$= 2\pi \left[\frac{1}{2} - \frac{2}{5} \right] = 2\pi \frac{1}{10}$$

$$\boxed{V = \frac{\pi}{5} //}$$

b-) around y-axis.

$$V = 2\pi \int_a^b x (g(x) - f(x)) dx.$$

$$= 2\pi \int_0^1 x (x^2 - 0) dx = 2\pi \int_0^1 x^3 dx$$

$$= 2\pi \left[\frac{x^4}{4} \right]_0^1 = 2\pi \frac{1}{4} = \frac{\pi}{2} //$$

Q7

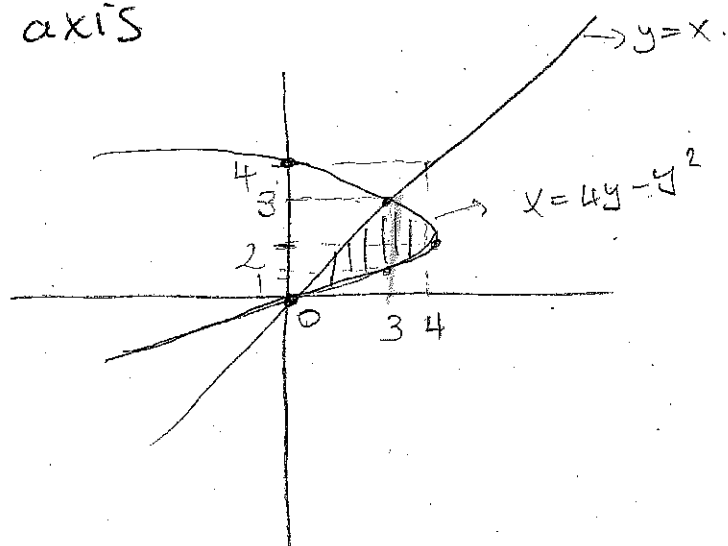
6

Ex: Find the volume of the solid obtained if region bounded by $y = x$ and $x = 4y - y^2$ are rotated about

a.) x-axis

b.) y-axis

Soln:



$$\begin{aligned} 0 &= 4y - y^2 \\ 0 &= y(4 - y) \\ y &= 0 \quad y = 4 \end{aligned}$$

$$\begin{aligned} y &= 4y - y^2 \\ 0 &= 3y - y^2 \\ 0 &= y(3 - y) \\ y &= 0 \quad y = 3 \end{aligned}$$

a.) about x-axis, by cylindrical shells.

$$V = 2\pi \int_c^d y(k(y) - h(y)) dy$$

$$= 2\pi \int_0^3 y(4y - y^2 - y) dy$$

$$= 2\pi \int_0^3 (3y^2 - y^3) dy = 2\pi \left[\frac{3y^3}{3} - \frac{y^4}{4} \right]_0^3$$

$$= 2\pi \left[27 - \frac{81}{4} \right] = \frac{27\pi}{2}$$

b.) about y-axis by ^{using} plane slices

$$V = \pi \int_c^d (k(y))^2 - (h(y))^2 dy$$

$$= \pi \int_0^3 [(4y - y^2)^2 - y^2] dy = \pi \int_0^3 (16y^2 - 8y^3 + y^4 - y^2) dy$$

$$= \pi \int_0^3 (15y^2 - 8y^3 + y^4) dy = \frac{108\pi}{5}$$

Sec 7.3Arc Length and Surface Area:

(7)

Defn: The arc length (length) of S of curve $y=f(x)$
 $x=f(y)$
from $x=a$ to $x=b$ is given by.

$$S = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Ex: Find the length of the curve $y = x^4 + \frac{1}{32x^2}$
from $x=1$ to $x=2$.

Soln: $\frac{dy}{dx} = 4x^3 - \frac{1}{16x^3}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(4x^3 - \frac{1}{16x^3}\right)^2$$

$$= 1 + (4x^3)^2 - 2(4x^3)\left(-\frac{1}{16x^3}\right) + \left(\frac{1}{16x^3}\right)^2$$

$$= \frac{1}{2} + (4x^3)^2 + \left(\frac{1}{16x^3}\right)^2$$

$$= \left(4x^3 + \frac{1}{16x^3}\right)^2$$

$$S = \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2} dx = \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left. x^4 - \frac{1}{32x^2} \right|_1^2$$

$$= 15.023 //$$

Area of a surface of Revolution

const by rotating arc length ds of the curve about given line.

If f' is continuous on $[a, b]$ and curve $y = f(x)$ is rotated about x-axis, then area of the surface of revolution so generated is

$$S = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2} dx$$

$$S = 2\pi \int_{x=a}^{x=b} |y| ds$$

if it is rotated about y-axis, surface area

$$S = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2} dx$$

If g' is continuous on $[a, d]$ and curve $x = g(y)$ is rotated about x-axis

$$\text{surface area} = S = 2\pi \int_c^d |y| \sqrt{1 + (g'(y))^2} dy$$

if rotation is about y-axis, surface area

$$S = 2\pi \int_c^d |g(y)| \sqrt{1 + (g'(y))^2} dy$$

(8)

Ex: Find the surface area of the shape obtained by rotating $y = x^2$ ($0 \leq x \leq 1$) about y-axis

Soln: $S = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2} dx$

$$f(x) = y = x^2$$

$$f'(x) = 2x$$

$$u = 1 + 4x^2$$

$$du = 8x dx$$

$$\frac{1}{8} du = x dx$$

$$S = 2\pi \int_0^1 x \sqrt{1 + 4x^2} dx$$

$$S = \frac{2\pi}{8} \int_1^5 \sqrt{u} du$$

$$= \frac{\pi}{6} \left(u^{3/2} \right) \Big|_1^5 = \frac{\pi}{6} (5\sqrt{5} - 1) //$$

$$\begin{aligned} x=0 & \quad u=1 \\ x=1 & \quad u=5 \end{aligned}$$

Q23

Ex: Find surface area of the shape obtained by rotating $y = x^{3/2}$ ($0 \leq x < 1$) about x-axis.

Soln: $S = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2} dx$

$$f(x) = y = x^{3/2}$$

$$f'(x) = 3x^{1/2}$$

$$u = 1 + 9x^2$$

$$du = 18x dx$$

$$= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^2} dx$$

$$= \frac{2\pi}{36} \int_1^{10} \sqrt{u} du = \frac{\pi}{18} \frac{2u^{3/2}}{3} \Big|_1^{10}$$

$$= \frac{\pi}{27} [10^{3/2} - 1] = \frac{\pi}{27} (10\sqrt{10} - 1)$$

Sec 8.2

Parametric Curves

object moving in xy -plane
and its coordinates at any time t
is a func of t .

Defn: A parametric curve C in the plane consists of an ordered pair (f, g) of continuous functions each defined on the same interval I .
path followed by moving object is a curve C .

$$x = f(t) \quad y = g(t) \quad t \in I$$

are called parametric eqns of the curve C .

Ex: Sketch and identify the parametric curve.

$$x = t^2 - 1 \quad y = t + 1$$

showing its direction with an arrow.

$(-\infty < t < \infty)$
Eliminate the parameter to
give a Cartesian eqn
in x and y whose
graph contains the parametric curve.

Soln:

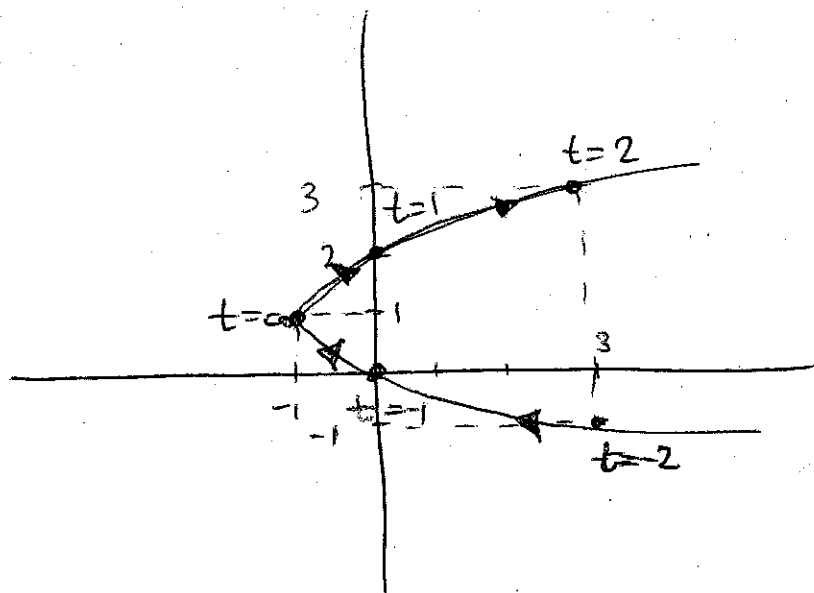
t	$x = t^2 - 1$	$y = t + 1$
-2	$x = 3$	$y = -1$
-1	$x = 0$	$y = 0$
0	$x = -1$	$y = 1$
1	$x = 0$	$y = 2$
2	$x = 3$	$y = 3$

$$x = t^2 - 1$$

$$t = 1 - y$$

$$x = (1 - y)^2 - 1 = 1 - 2y + y^2 - 1$$

$x = y^2 - 2y \rightarrow$ all points on the curve lie on this parabola.



parabola defined
parametrically

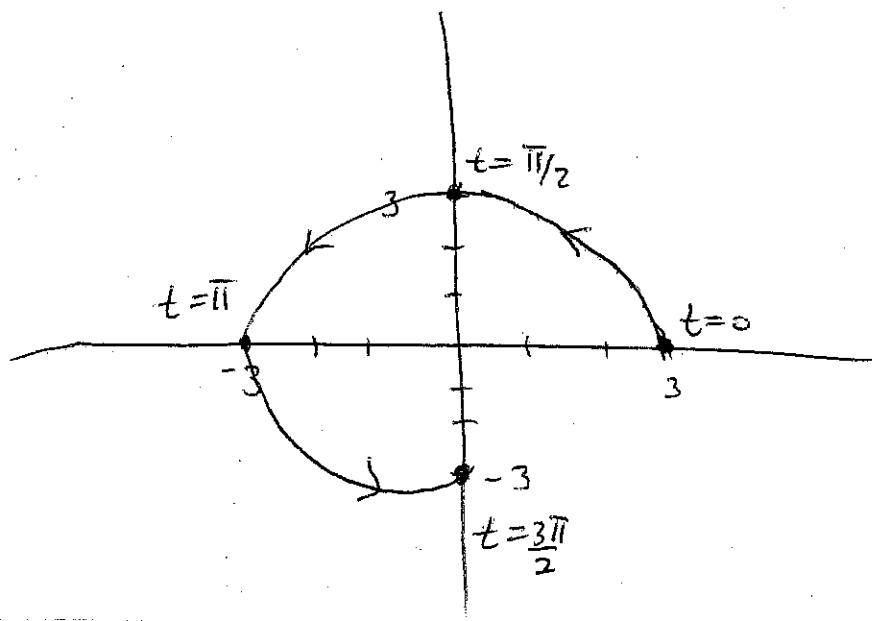
↓
Ex; sketch and identify the curve $x = 3 \cos t$, $y = 3 \sin t$
($0 \leq t \leq \frac{3\pi}{2}$)

t	$x = 3 \cos t$	$y = 3 \sin t$
0	$x = 3$	$y = 0$
$\frac{\pi}{2}$	$x = 0$	$y = 3$
π	$x = -3$	$y = 0$
$\frac{3\pi}{2}$	$x = 0$	$y = -3$

$$x^2 + y^2 = 9 \cos^2 t + 9 \sin^2 t$$

$$= 9 (\cos^2 t + \sin^2 t)$$

$$x^2 + y^2 = 9$$



Ex: Sketch the given parametric curve, showing its direction.

$$x = 3 \sin \pi t \quad y = 4 \cos \pi t \quad (-1 \leq t \leq 1)$$

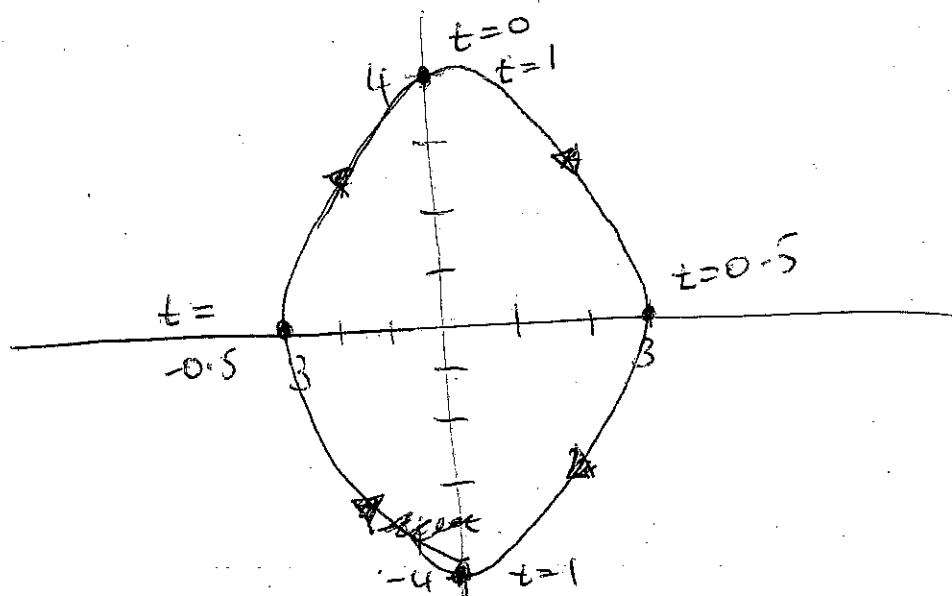
Soln:

t	$x = 3 \sin \pi t$	$y = 4 \cos \pi t$
-1	$x = 0$	$y = -4$
-0.5	$x = -3$	$y = 0$
0	$x = 0$	$y = 4$
0.5	$x = 3$	$y = 0$
1	$x = 0$	$y = -4$

$$x^2 + y^2 = 9 \sin^2 \pi t + 16 \cos^2 \pi t$$

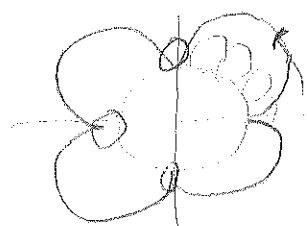
$$\frac{x}{3} = \sin \pi t \quad \frac{y}{4} = \cos \pi t$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \sin^2 \pi t + \cos^2 \pi t \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1$$



Ex: Find the arclength of the parametric curve

$$x = 5 \cos t - \sin t \quad \text{and} \quad y = 5 \sin t - \cos t \quad 0 \leq t \leq \pi/2$$



$$S = 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 4 \int_0^{\pi/2} \sqrt{(5 \sin t + \sin t)^2 + (5 \cos t - \cos t)^2} dt$$

$$= 20 \int_0^{\pi/2} \sqrt{2 - 2 \sin t \cos t - 2 \cos t \sin t} dt$$

$$= 20 \int_0^{\pi/2} \sqrt{2 - 2 \cos(4t)} dt$$

$$= 20 \int_0^{\pi/2} \sqrt{4 \sin^2 2t} dt = 40 \int_0^{\pi/2} \sin 2t dt = -20 [\cos 2t]_0^{\pi/2} = 40$$

Sec 8.4 Arc Lengths and Areas of Parametric Curves.

Let C be a parametric curve with eqns $x=f(t)$ $y=g(t)$
 $a \leq t \leq b$.

Defn: The arc length element for a parametric curve.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Defn: The length of curve C .

$$S = \int_{t=a}^{t=b} ds = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Ex: Find the length of the parametric curve.

$$x = e^t \cos t \quad y = e^t \sin t \quad 0 \leq t \leq 2.$$

Soln:

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t = e^t (\cos t - \sin t)$$

$$\frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t (\sin t + \cos t)$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= e^{2t} (\cos t - \sin t)^2 + e^{2t} (\sin t + \cos t)^2$$

$$\frac{ds}{dt} = \sqrt{2e^{2t}}$$

length of the curve,

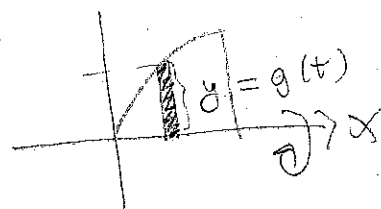
$$S = \int_0^2 \sqrt{2e^{2t}} dt = \sqrt{2} \int_0^2 e^t dt = \sqrt{2} (e^2 - 1) //$$

Surface Area of Parametric Curves

If parametric curve with eqns

$$x = f(t), \quad y = g(t) \quad a \leq t \leq b$$

is rotated about x-axis,

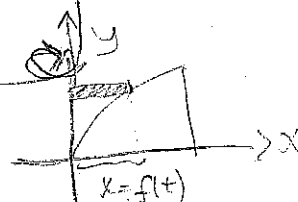


the area S of surface so generated is.

$$S = 2\pi \int_a^b |g(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

if rotation is about y-axis

$$S = 2\pi \int_a^b |f(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



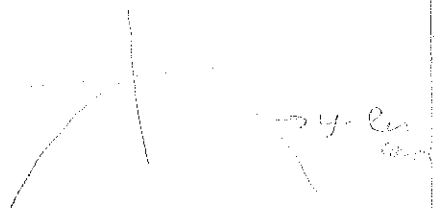
↓
Ex: Find the area of the surface of revolution obtained by rotating the curve

$$x = a \cos^3 t, \quad y = a \sin^3 t \quad (a > 0) \text{ about } \underline{\underline{x\text{-axis}}}$$

$$0 \leq t \leq \frac{\pi}{2}$$

1) Find the arclength of the graph of $y = \ln(\cos x)$ from $x=0$ to $x=\pi/4$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$



$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2} dx = \int_0^{\pi/4} \sqrt{\sec^2 x} = \int_0^{\pi/4} \sec x dx$$

$$= \ln |\sec x + \tan x| \Big|_0^{\pi/4} = \ln(\sqrt{2}+1) - \ln(1) = 0.881$$

2. Find the arclength of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on $[\frac{1}{2}, 2]$

$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2} \left(x^2 - \frac{1}{x^2} \right)$$

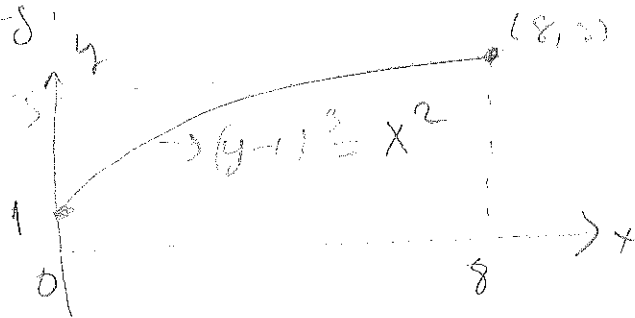
$$s = \int_{1/2}^2 \sqrt{1 + \left(\frac{1}{2} (x^2 - \frac{1}{x^2}) \right)^2} dx = \int_{1/2}^2 \sqrt{\frac{1}{4} \left(x^4 + 2 + \frac{1}{x^4} \right)}$$

$$= \int_{1/2}^2 \frac{1}{2} \left(x^2 + \frac{1}{x^2} \right) dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{1}{x} \right]_{1/2}^2$$

$$= \frac{1}{2} \left[\frac{8}{3} - \frac{4}{2} + \frac{1}{2} - \frac{2}{1} \right]$$

$$= \frac{33}{16}$$

3. Find the area of the surface $(y-1)^3 = x^2$ on $[4, 8]$ as shown in fig.



$$\frac{dx}{dy} = \frac{1}{2} (y-1)^{1/2}$$

$$S = \int_1^5 \sqrt{1 + \left(\frac{1}{2} (y-1)^{1/2}\right)^2} dy = \int_1^5 \sqrt{\frac{5y-5}{2}} dy$$

$$= \frac{1}{2} \int_1^5 \sqrt{5y-5} dy = \frac{1}{18} \frac{(5y-5)^{3/2}}{3/2} \Big|_1^5$$

$$= \frac{1}{27} [(40)^{3/2} - (4)^{3/2}] \approx 9.073$$

911

(12 b)

Ex: Find the area of the surface generated by rotating the curve $x = e^t \cos t$ $y = e^t \sin t$ ($0 \leq t \leq \frac{\pi}{2}$)

about x-axis.

Soln:
$$S = 2\pi \int_0^{\pi/2} e^t \sin t \cdot ds = 2\pi \int_0^{\pi/2} e^t \sin t \sqrt{2e^{2t}} dt$$

$$= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \sin t dt$$

$$u = e^{2t} \quad dv = \sin t$$

$$du = 2e^{2t} dt \quad v = -\cos t$$

$$\int_0^{\pi} e^{2t} \sin t dt = \left[\cos t e^{2t} \Big|_0^{\pi/2} + 2 \int_0^{\pi/2} e^{2t} \cos t dt \right]$$

$$u = e^{2t}$$

$$du = 2e^{2t} dt$$

$$dv = \cos t dt$$

$$v = \sin t$$

$$= \left[0 + 1 + 2 \left(\frac{e^{2t} \sin t}{2} \Big|_0^{\pi/2} - 2 \int_0^{\pi/2} e^{2t} \sin t dt \right) \right]$$

$$= \left[1 + 2 \left[e^{\pi} - 0 \right] - 4 \int_0^{\pi/2} e^{2t} \sin t dt \right]$$

$$\int_0^{\pi} e^{2t} \sin t dt = \frac{1 + 2e^{\pi}}{5}$$

$$S = 2\sqrt{2}\pi \left(\frac{1 + 2e^{\pi}}{5} \right)$$

Soln: $S = 2\pi \int_a^b |g(t)| \sqrt{(f'(t))^2 + (g'(t))^2} dt$

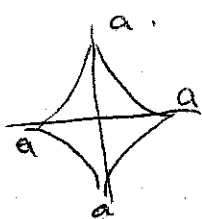
$$\frac{dx}{dt} = -3a \cos^2 t \sin t \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\frac{ds}{dt} = ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$ds = \sqrt{9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t} dt$$

$$ds = 3a \cos t \sin t \sqrt{\underbrace{\cos^2 t + \sin^2 t}_1} dt$$

$$ds = 3a \cos t \sin t dt$$



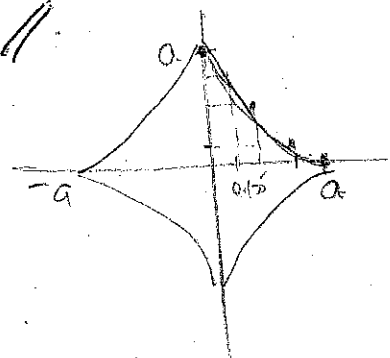
entire surface
is generated by
rotating upper half
of the curve

$$S = 2\pi \times 2 \int_0^{\pi/2} a \sin^3 t \cdot 3a \cos t \sin t dt$$

$$S = 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t dt$$

$$u = \sin t \quad = 12\pi a^2 \int u^4 \cdot du$$

$$du = \cos t dt \quad = \frac{u^5}{5} = \frac{12\pi a^2 \sin^5 t}{5} \Big|_0^{\pi/2} = \frac{12\pi a^2}{5} //$$



$$x = a(\cos t)^3$$

$$t=0$$

$$y = a$$

$$t = \frac{\pi}{6}$$

$$x = (0.86)^3 a = 0.63a$$

$$t = \frac{\pi}{4}$$

$$x = (0.70)^3 a = 0.34a$$

$$t = \frac{\pi}{3}$$

$$x = (0.5)^3 a = 0.125a$$

$$t = \frac{\pi}{2}$$

$$x = 0$$

$$y = a(\sin t)^3$$

$$y = 0 = 0$$

$$y = (0.5)^3 a = 0.125a$$

$$y = (0.70)^3 a = 0.34a$$

$$y = (0.86)^3 a = 0.63a$$

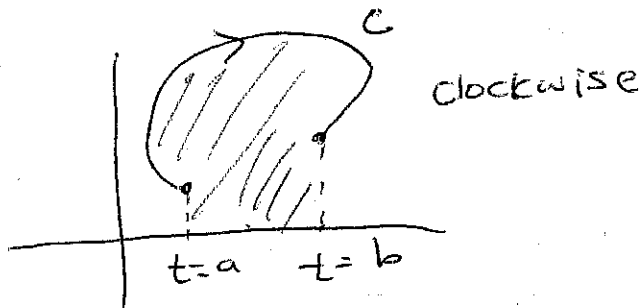
$$y = a$$

Areas Bounded By Parametric Curves

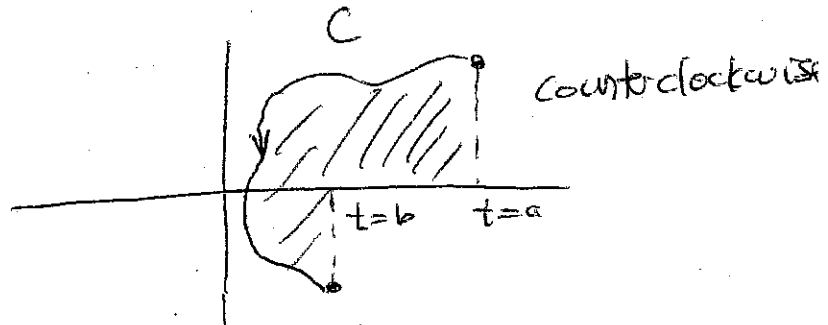
13

Consider parametric curve C with eqns

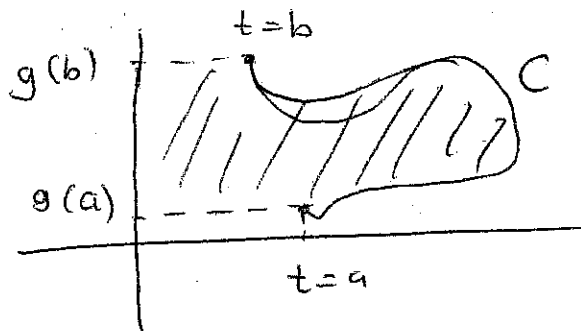
$$\boxed{x = f(t), \quad y = g(t) \quad (a \leq t \leq b)}$$



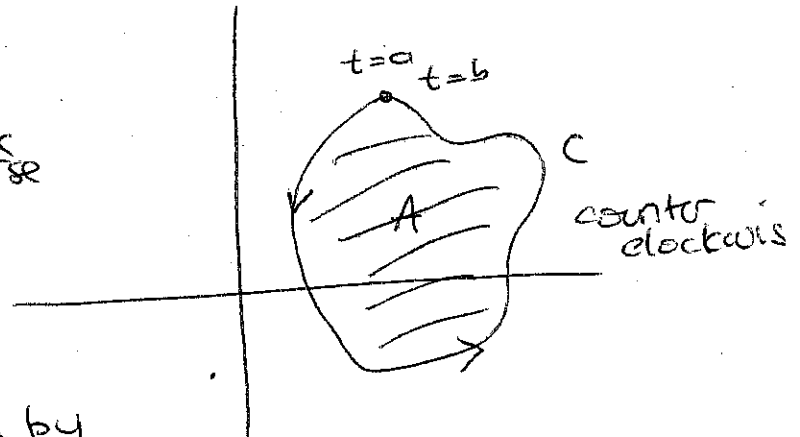
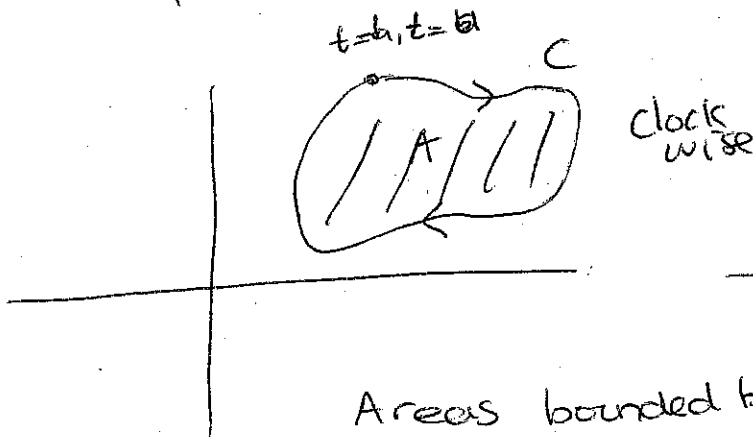
$$A = \int_a^b g(t) \cdot f'(t) dt$$



$$A = - \int_a^b g(t) f'(t) dt$$



$$\rightarrow \boxed{A = \int_a^b f(t) g'(t) dt}$$



Areas bounded by
closed parametric curves.

$$\boxed{A = \int_a^b g(t) f'(t) dt}$$

clockwise

$$\boxed{A = - \int_a^b g(t) f'(t) dt}$$

counterclockwise

Area
bounded by parametric curve,

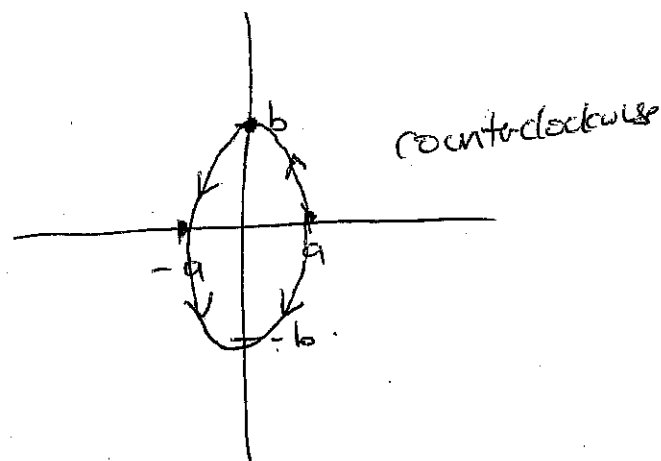
(14) $f(t)$

Ex: Find the area bounded by the ellipse $x = a \cos s$,

$$y = b \sin s \quad (0 \leq s \leq 2\pi)$$

$f(t)$

s	$x = a \cos s$	$y = b \sin s$
0	$x = a$	$y = 0$
$\frac{\pi}{2}$	$x = 0$	$y = b$
π	$x = -a$	$y = 0$
$\frac{3\pi}{2}$	$x = 0$	$y = -b$
2π	$x = a$	$y = 0$



$$A = - \int_0^{2\pi} g(t) \cdot f'(t) dt = - \int_0^{2\pi} b \sin s (-a \sin s) ds$$

$$= ab \int_0^{2\pi} \sin^2 s ds = \frac{ab}{2} \int_0^{2\pi} (1 - \cos 2s) ds$$

$$= \frac{ab}{2} \left[(2\pi - 0) + \underbrace{\frac{1}{2} \sin 2s}_0 \right] = \pi ab //$$

Sec 8.5 Polar Coordinates & Polar Curves

In polar coordinates

- origin (POLE) is O ,

- polar axis = a ray extending from O horizontally to the right.

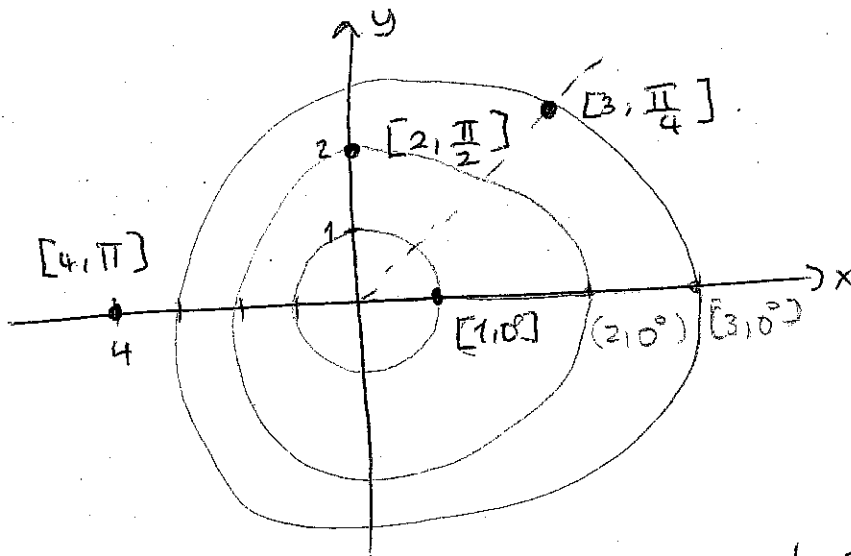
any point P in polar coordinates is determined by

$[r, \theta]$

(i) r = distance from O to P

(ii) θ = angle that the ray OP makes with polar axis (counterclockwise angles are positive).

Ex:



* $[r, \theta_1]$ and $[r, \theta_2] \rightarrow$ represent same point

If $\theta_2 = \theta_1 + 2n\pi$ $n = 0, \pm 1, \pm 2, \dots$ $\frac{\pi}{2} - \frac{4\pi}{2} = -$

Ex: $[1, \frac{\pi}{2}]$, $[1, \frac{5\pi}{2}]$, $[1, -\frac{3\pi}{2}] \rightarrow$ all represent same point.

* $[r, \theta] = [-r, \theta + \pi]$

$$\theta + \pi = \frac{\pi}{4} \quad \theta = \frac{\pi}{4} \quad \frac{4\pi}{4}$$

Ex: $[-1, \frac{\pi}{4}] = [1, \frac{5\pi}{4}]$

Relation between polar & Cartesian coord:

$x = r \cos \theta$	$x^2 + y^2 = r^2$
$y = r \sin \theta$	$\tan \theta = \frac{y}{x}$

Ex: Find the Cartesian eqn of the curve represented by polar eqn $r = 2a \cos \theta$ and hence identify the curve.

Soln: $x = r \cos \theta = 2a \cos^2 \theta$

$$y = r \sin \theta = 2a \cos \theta \sin \theta$$

$$\begin{aligned} x^2 + y^2 &= 4a^2 (\cos^2 \theta)^2 + 4a^2 \cos^2 \theta \sin^2 \theta \\ &= 4a^2 \cos^2 \theta \left[\underbrace{\cos^2 \theta + \sin^2 \theta}_1 \right] \end{aligned}$$

$$x^2 + y^2 = 2a \cdot \underbrace{(2a \cos^2 \theta)}_x$$

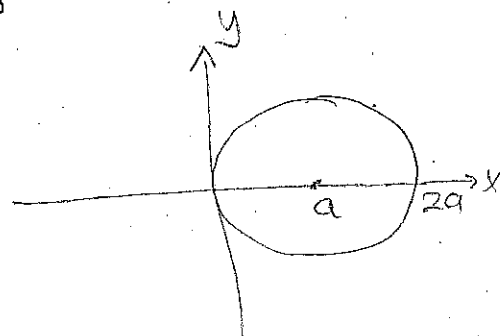
$$x^2 + y^2 = 2ax$$

$$x^2 - 2ax + y^2 = 0$$

$$(x-a)^2 - a^2 + y^2 = 0$$

$$(x-a)^2 + y^2 = a^2$$

circle with centre $(a, 0)$
radius = a .



Ex: Find the polar coordinates of the straight line
 $2x - 3y = 5$.

Soln:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

$$2(r \cos \theta) - 3r \sin \theta = 5$$

$$r(2 \cos \theta - 3 \sin \theta) = 5 \Rightarrow r = \frac{5}{2 \cos \theta - 3 \sin \theta}$$

Q1.

Ex: Transform the given polar eqn to rectangular coordinates and hence identify the curve represented.

a.) $r = 3 \sec \theta$

Soln: $r = \frac{3}{\cos \theta} \Rightarrow r \cos \theta = 3$

$x = 3$ straight vertical line.

^{Q2}
b.) $r = \sec \theta (1 + \tan \theta)$

soln: $r^2 = r \sec \theta (1 + \tan \theta)$

$$r^2 = \frac{r}{\cos \theta} + \frac{r \sin \theta}{\cos^2 \theta} = \frac{r \cos \theta + r \sin \theta}{\cos^2 \theta}$$

$$r^2 \cos^2 \theta = r \cos \theta + r \sin \theta$$

$$x^2 = x + y \Rightarrow \boxed{y = x^2 - x} \text{ parabola.}$$

Some Polar Curves :

Ex: Sketch and identify the curve $r = 2a \cos(\theta - \theta_0)$.

Soln: $r^2 = 2ar \cos(\theta - \theta_0) = 2ar \cos\theta \cos\theta_0 + 2ar \sin\theta \sin\theta_0$

$$x^2 + y^2 = 2ax \cos\theta_0 + 2ay \sin\theta_0.$$

$$x^2 - 2ax \cos\theta_0 + a^2 \cos^2\theta_0 + y^2 - 2ay \sin\theta_0 + a^2 \sin^2\theta_0 = a^2.$$

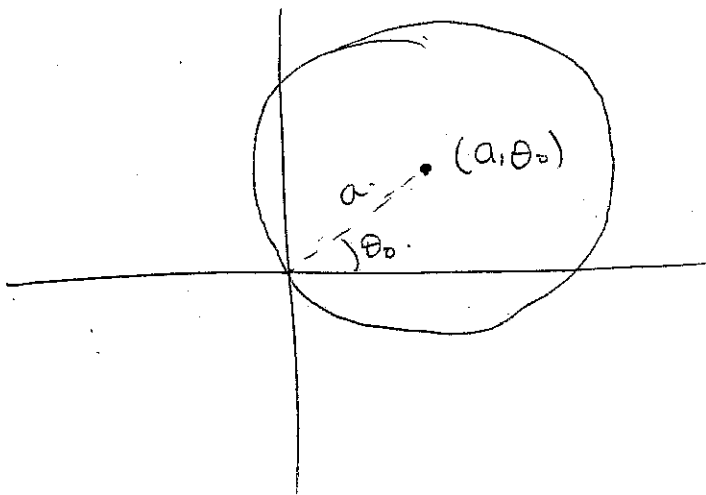
$$(x - a \cos\theta_0)^2 + (y - a \sin\theta_0)^2 = a^2$$

$$\Rightarrow x = a \cos\theta_0 \quad \Rightarrow y = a \sin\theta_0$$

circle of radius = a its center: $(a \cos\theta_0, a \sin\theta_0)$
in cartesian coord.

$$r^2 = a^2 \cos^2\theta_0 + a^2 \sin^2\theta_0 \Rightarrow r = a$$

in polar: (a, θ_0) .



it passes through the origin in directions $\theta = \theta_0 \pm \frac{\pi}{2}$. make $r=0$

Ex: Sketch the polar curve $r = a(1 - \cos\theta)$ $a > 0$.

Soln: In rectangular coord.

$$r^2 = ar - a r \cos\theta$$

$$x^2 + y^2 = a \sqrt{x^2 + y^2} - ax$$

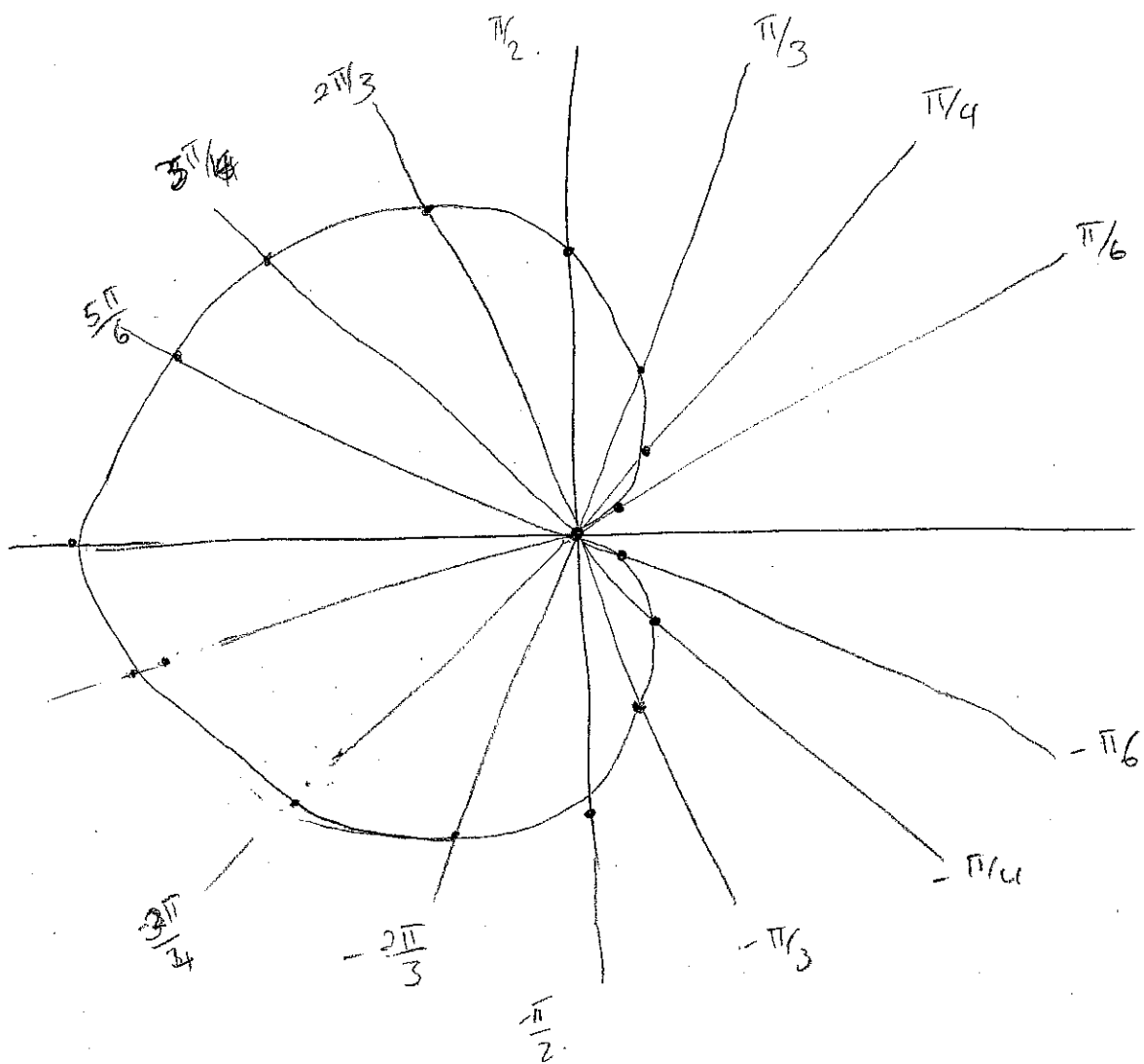
$$x^2 + y^2 + ax = a \sqrt{x^2 + y^2}$$

$$(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$$

→ we don't recognize this curve

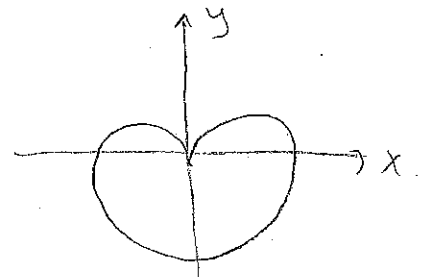
So, make table of values:

θ	$r = a - a \cos\theta = a(1 - \cos\theta)$
0	$r = 0$
$\mp \frac{\pi}{6}$	$r = a(1 - \frac{\sqrt{3}}{2}) = 0.13a$
$\mp \frac{\pi}{4}$	$r = a(1 - \frac{\sqrt{2}}{2}) = 0.29a$
$\mp \frac{\pi}{3}$	$r = a(1 - \frac{1}{2}) = 0.5a$
$\mp \frac{\pi}{2}$	$r = a(1 - 0) = a$
120° $\mp \frac{2\pi}{3}$	$r = a(1 + 0.5) = 1.5a$
135° $\mp \frac{3\pi}{4}$	$r = a(1 + \frac{\sqrt{2}}{2}) = 1.71a$
150° $\mp \frac{5\pi}{6}$	$r = a(1 + 0.87) = 1.87a$
π	$r = a(1 + 1) = 2a$



The cardioid.

Exercise: sketch $r = a(1 - \sin \theta)$



Ex: Sketch the polar graphs

a.) $r = \cos 2\theta$

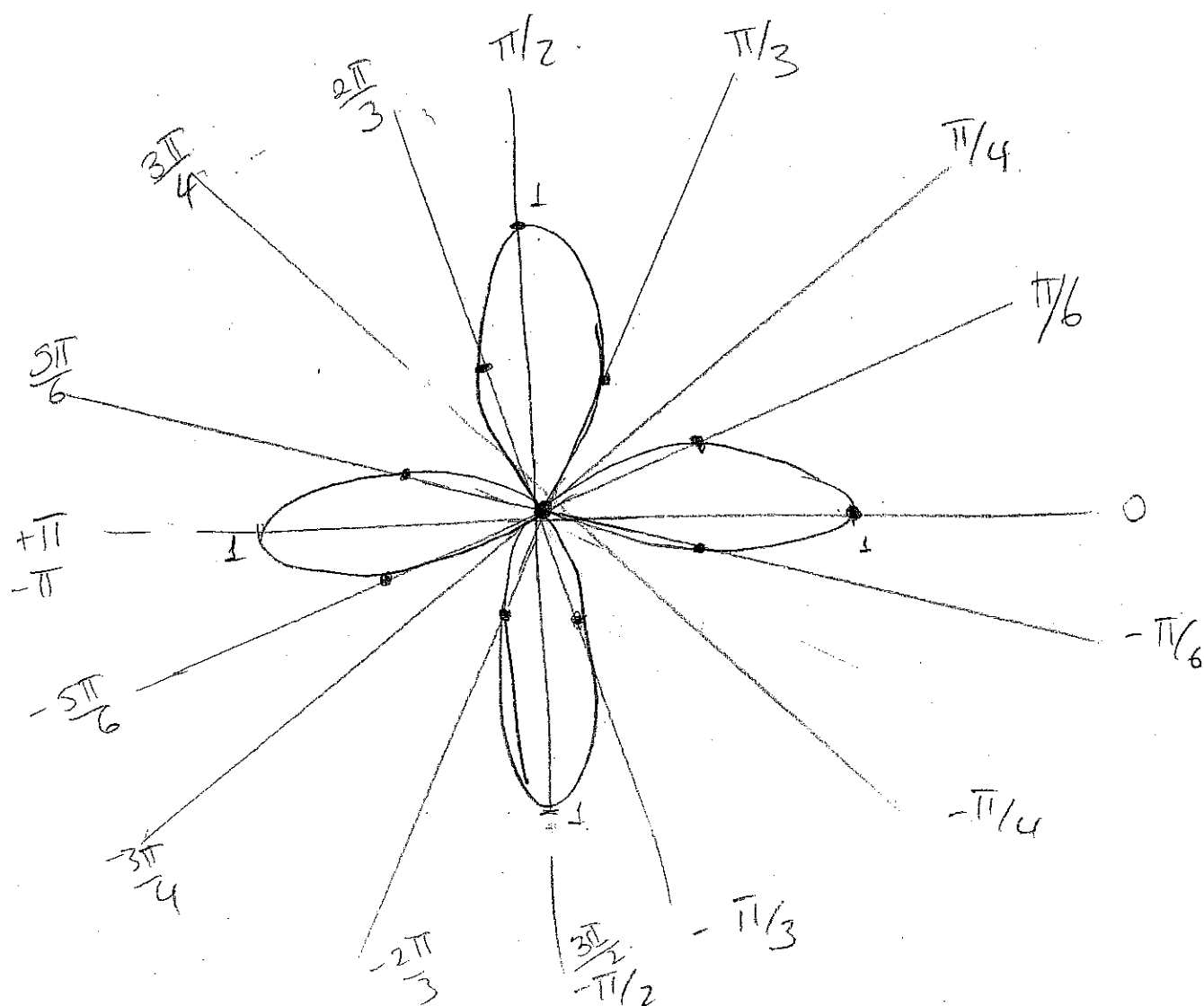
b.) $r = \sin 3\theta$

c.) $r^2 = \cos 2\theta$

θ	$r = \cos 2\theta$
0	$r = 1$
$\mp \frac{\pi}{6}$	$r = \cos\left(\mp \frac{\pi}{3}\right) = 0.5$
$\mp \frac{\pi}{4}$	$r = \cos\left(\mp \frac{\pi}{2}\right) = 0$
$\mp \frac{\pi}{3}$	$r = \cos\left(\mp \frac{2\pi}{3}\right) = -0.5$
$\mp \frac{\pi}{2}$	$r = \cos(\mp \pi) = -1$
$\mp \frac{2\pi}{3}$	$r = \cos\left(\mp \frac{4\pi}{3}\right) = -0.5$
$\mp \frac{3\pi}{4}$	$r = \cos\left(\mp \frac{3\pi}{2}\right) = 0$
$\mp \frac{5\pi}{6}$	$r = \cos\left(\mp \frac{5\pi}{3}\right) = 0.5$
$\mp \pi$	$r = \cos(\mp 2\pi) = 1$

$$[r, \theta] = [-r, \theta + \pi]$$

$$\left[-1, \frac{\pi}{2}\right] = [-1, \frac{\pi}{2} + \pi] = \left[1, \frac{3\pi}{2}\right]$$



Sec 8.6 Slopes, Areas and Arc Lengths for Polar Curves

Areas Bounded By Polar Curves:

Defn: Area in polar Coordinates

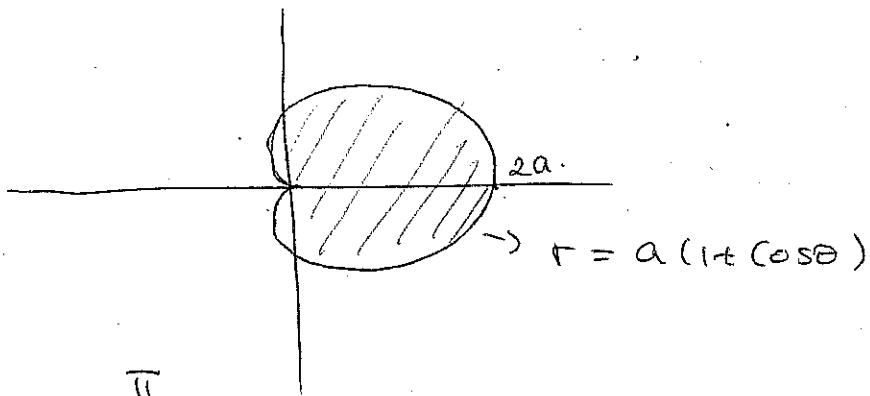
The region bounded by $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ ($\alpha < \beta$) has area,

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta$$

Ex: Find the area bounded by the cardioid

$$r = a(1 + \cos \theta)$$

Soln:



$$\text{Area} = 2 \cdot \frac{1}{2} \int_0^{\pi} a^2 (1 + \cos \theta)^2 d\theta$$

$$= a^2 \int_0^{\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta$$

$$= a^2 \left[\int_0^{\pi} 1 d\theta + 2 \int_0^{\pi} \cos \theta d\theta + 2 \int_0^{\pi} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \right]$$

$$= \frac{3}{2} \pi a^2 //$$

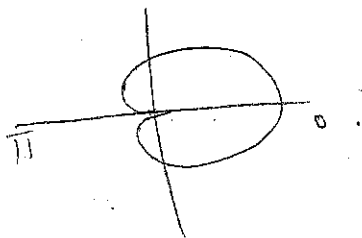
Defn: Arc Length element for polar curve:

The arc length element for the polar curve $r = f(\theta)$ is

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta = \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta$$

$$\text{length} = \int_a^b ds$$

Ex: Find the total length of the cardioid $r = a(1 + \cos\theta)$.



$$S = 2 \int_0^{\pi} ds$$

$$\frac{dr}{d\theta} = -a \sin\theta$$

$$S = 2 \int_0^{\pi} \sqrt{a^2 \sin^2\theta + a^2 (1 + \cos\theta)^2} d\theta$$

$a^2(\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta)$
 $a^2(2 + 2\cos\theta)$

$$= 2 \int_0^{\pi} \sqrt{\frac{2a^2 + 2a^2 \cos\theta}{2a^2(1 + \cos\theta)}} d\theta$$

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

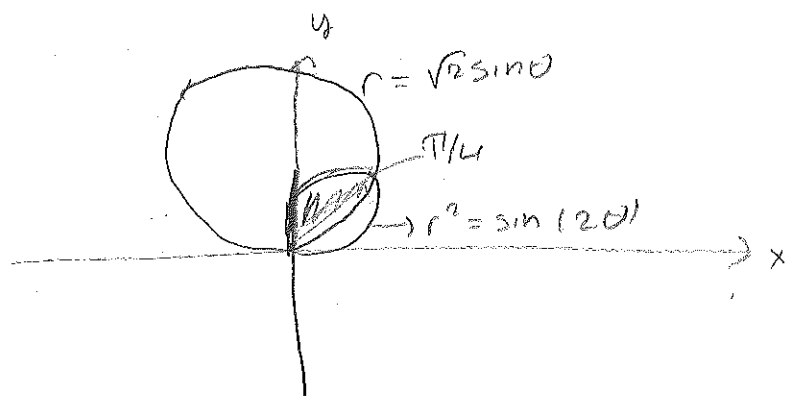
$$= 2\sqrt{2}a \int_0^{\pi} \sqrt{\frac{2\cos^2\frac{\theta}{2}}{2}} d\theta$$

$$= 4a \int_0^{\pi} \cos\frac{\theta}{2} d\theta = 8a \sin\frac{\theta}{2} \Big|_0^{\pi} = 8a$$

$$2\cos^2\theta = \cos 2\theta + 1$$

$$2\cos^2\frac{\theta}{2} = \cos\theta + 1$$

Find the area of the region that lies inside the circle $r = \sqrt{2} \sin \theta$ and inside the lemniscate $r^2 = \sin 2\theta$



$$2 \sin^2 \theta = \sin 2\theta$$

$$2 \sin^2 \theta = 2 \sin \theta \cos \theta$$

$$\sin^2 \theta - \sin \theta \cos \theta = 0$$

$$\sin \theta (\sin \theta - \cos \theta) = 0$$

$$\sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$$

$$\sin \theta - \cos \theta = 0$$

$$\sin \theta = \cos \theta$$

$$\theta = \pi/4$$

$$A = \frac{1}{2} \int_0^{\pi/4} 2 \sin^2 \theta d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \sin 2\theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{1 - \cos 2\theta}{2} d\theta - \frac{1}{4} \cos 2\theta \Big|_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2} \theta - \frac{\sin 2\theta}{4} \Big|_0^{\pi/4} + \frac{1}{4}$$

$$= \frac{\pi}{8} - \frac{1}{4} + \frac{1}{4}$$

$$= \frac{\pi}{8}$$

Find the area of one petal of the rose curve given by

$$r = 3 \cos 3\theta$$

$$A = \frac{1}{2} \int_{-\pi/6}^{\pi/6} r^2 d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/6} (3 \cos 3\theta)^2 d\theta$$

$$= \frac{9}{2} \int_{-\pi/6}^{\pi/6} \cos^2 3\theta d\theta$$



0
 $\pi/6$

1
0

YAKIN DOĞU ÜNİVERSİTESİ EĞİTİM FAKÜLTESİ

MATEMATİK GÜNLERİ

PROGRAM

Tarih: 11/03/2014

Saat: 10:30 Yer: 11D29

- Eğitim Fakültesi Dekanı Prof.Dr. Hüseyin Uzunboylu'nun konuşması.
- Fen ve Matematik Alanları Eğitimi Bölüm Başkanı
Yrd.Doç.Dr. Murat Tezer'in Konuşması.
- Matematik Eğitimi Kulüp Başkanı Algi Evrener'in Konuşması.
- Matematik Bölüm Başkanı Prof.Dr. Kaya Özkın'ın Semineri
Sayılar

Saat: 14:00 Yer: 11D22

- Bilgisayar ve Öğretim Teknolojileri Bölüm Başkanı Yrd.Doç.Dr. Fezile Özdamlı'nın Semineri
Teknoloji ve Matematik

Tarih:12/03/2014

Saat: 15:00 Yer: SOS çocuk köyü

- SOS Çocuk köyünde kimsesiz çocuklara yönelik **Matematik Etütünün** Matematik Eğitimi Kulübünce başlatılması.

Tarih: 13/03/2014

Saat: 14:00 Yer: 11D22

- Fen ve Matematik Alanları Eğitimi Bölüm Başkanı Yrd.Doç.Dr. Murat Tezer'in Semineri
Matematik Çalışma Becerileri ve 21. Yüzyılın Matematik Öğretmeni

Tarih: 14/03/2014

Saat: 14:30 Yer: 11D22

- Pi Gününün Kutlanması (Saat: 3,14)
- KAPANIŞ

Find the length of the arc from $\theta=0$ to $\theta=2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$

$$S = \int_a^b \sqrt{f(\theta)^2 + (f'(\theta))^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2} d\theta$$

$$4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta$$

$$= 2\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\theta} d\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$$

$$= 2\sqrt{2} \int_0^{2\pi} \sqrt{1 - 1 + 2\sin^2\frac{\theta}{2}} d\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$= 4 \int_0^{2\pi} \sin\frac{\theta}{2} d\theta$$

$$= 4 \cdot 2 \left[-\cos\frac{\theta}{2} \right]_0^{2\pi}$$

$$= 8(1+1) = 16$$

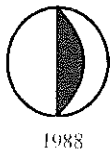
$$= \frac{1}{2} \int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2 d\theta + \frac{1}{2} \int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2 d\theta$$

$$= 18 \int_{\pi/2}^{2\pi/3} \cos^2\theta d\theta + \frac{1}{2} \int_{2\pi/3}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta) d\theta$$

$$= 9 \int_{\pi/2}^{2\pi/3} (1 + \cos 2\theta) d\theta + \int_{2\pi/3}^{\pi} (3 - 4\cos\theta + \cos 2\theta) d\theta$$

$$= 9 \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/2}^{2\pi/3} + \left(3\theta - 4\sin\theta + \frac{\sin 2\theta}{2} \right)_{2\pi/3}^{\pi}$$

$$= 9 \left(\frac{2\pi}{3} - \frac{\pi}{2} - \frac{3}{4} \right) + (3\pi - 2\pi + 2\sqrt{3} + \frac{1}{2}) = \frac{5\pi}{2} \approx 7.85$$



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Department of Mathematics

1988

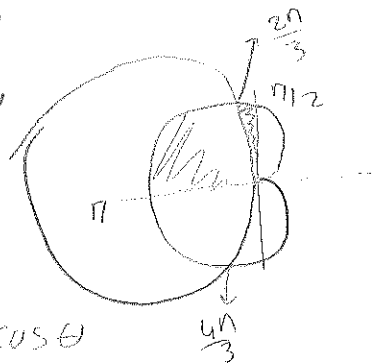
$$\text{Area} \\ r = -6\cos\theta$$

$$r = 2 - 2\cos\theta$$

$$-6\cos\theta = 2 - 2\cos\theta$$

$$-2 = 4\cos\theta$$

$$\frac{1}{2} \cos\theta$$



$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Name-Surname : Bilgen Kaymakmazade

Student Number : 20061883

Registration Date : 2011-2012 Fall

Status : Master with Thesis

2011-2012 Fall Semester:

	Course Code	Course Name	Grade
1	M 511	Real Analysis I	AA
2	M 555	Complex Analysis I	AA
CGPA:			4,00

2011-2012 Spring Semester:

	Course Code	Course Name	Grade
1	M 556	Complex Analysis II	AA
2	M 598	Research Methods	BA
3	M 512	Real Analysis I	AA
CGPA:			3,83

2012-2013 Fall Semester:

	Course Code	Course Name	Grade
1	M 503	Functional Analysis I	BA
2	M 523	Algebra I	AA
3.	M 599	Seminar	S
CGPA:			3,75

2012-2013 Spring Semester:

	Course Code	Course Name	Grade
1	M 600	Thesis	S
GPA:			3,85

2013-2014 Fall Semester:

	Course Code	Course Name	Grade
1	M 600	Thesis	S
GPA:			3,85
Starting Semestre	Advisor	Thesis Title	Finishing Date
2012-13 Spring	Prof. Dr. İ.Kaya Özkin	Hyperbolic Geometry And Complex Analysis	January 2014

$X^2 + 5x + 6 = 0$