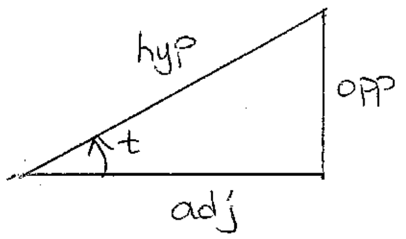
P.6 The Trigonometric Functions

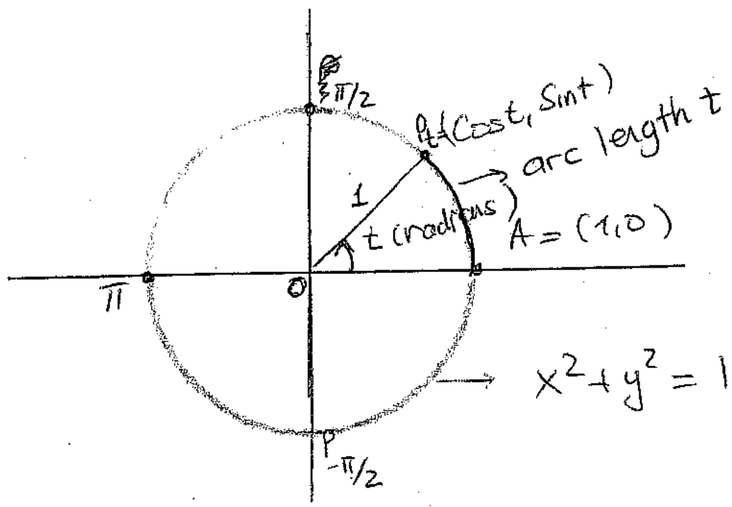


$\cos t = \frac{adj}{hyp}$

$\sin t = \frac{opp}{hyp}$

No.

Defn: Radian measure of angle. ($A O P_t$).

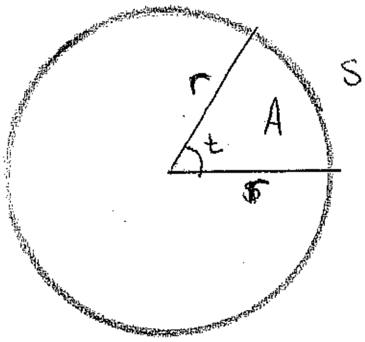


$\angle AOP_t$ = t radians.

$\boxed{\pi \text{ radians} = 180^\circ}$

* To convert degrees to radians multip by $\frac{\pi}{180}$
* To " radians → degrees " " $\frac{180}{\pi}$

Ex:



$$\text{Arc length} = s = \frac{t}{2\pi}(2\pi r) = \underline{\underline{rt}} \text{ unit}$$

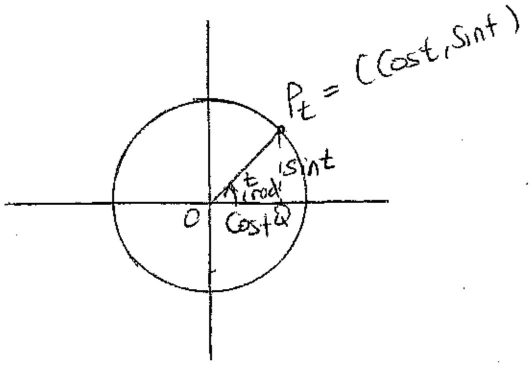
$$\text{Sector Area} = \frac{t}{2\pi}(\pi r^2) = \underline{\underline{\frac{r^2 t}{2}}} \text{ units}$$

Defn:

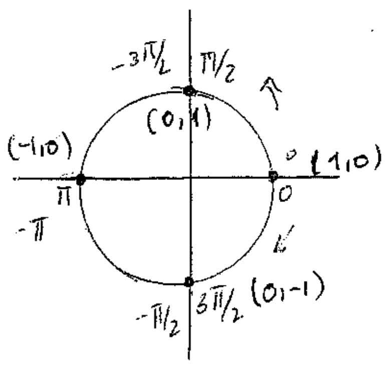
For any real t,

$\cos t$ = the x-coordinate of P_t

$\sin t$ = the y-coordinate of P_t



Ex:



$\cos 0 = 1$

$\sin 0 = 0$

$\cos \frac{\pi}{2} = 0$

$\sin \frac{\pi}{2} = 1$

$\cos(-\pi/2) = \cos \frac{3\pi}{2} = 0$

$\sin(-\pi/2) = \sin \frac{3\pi}{2} = -1$

$\cos \pi = -1$

$\sin \pi = 0$

Some Useful Identities :

The range of cosine and sine :

$$-1 \le \cos t \le 1 \quad , \quad -1 \le \sin t \le 1$$

(1.) $\cos^2 t + \sin^2 t = 1$ → Pythagorean Identity.

(2.) $\cos(t+2\pi) = \cos t$

$\sin(t+2\pi) = \sin t$

} Periodicity. Cosine & Sine functions are periodic with period 2π

(3.)

$\cos(-t) = \cos t$ → Cosine is even function.

$\sin(-t) = -\sin t$ → Sine is an odd funct.

(4.) Complementary angle identities :

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \quad , \quad \sin\left(\frac{\pi}{2} - t\right) = \cos t.$$

(5.) Supplementary angle identities :

$$\cos(\pi - t) = -\cos t \quad , \quad \sin(\pi - t) = \sin t.$$

Some Special Angles:

Degrees	0°	30°	45°	60°	90°	135°	180°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{2}}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1

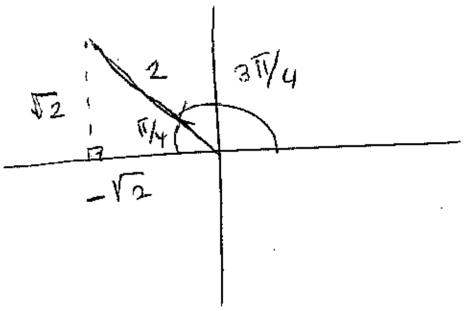
$$\cos 120^\circ = \cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\cos$$

$$= -\frac{1}{2}$$

Ex: Find,

(a.) $\sin\left(\frac{3\pi}{4}\right)$

$$\sin(\pi - t) = \sin t$$



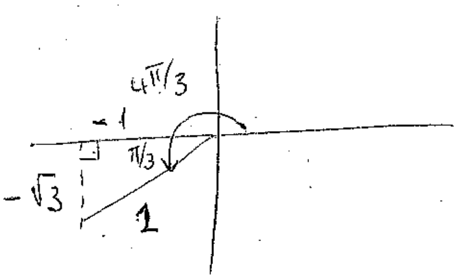
$$\sin\left(\frac{3\pi}{4}\right) = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}$$

$$\cos(\pi - t) = -\cos t$$

(b.) $\cos\left(\frac{4\pi}{3}\right) =$

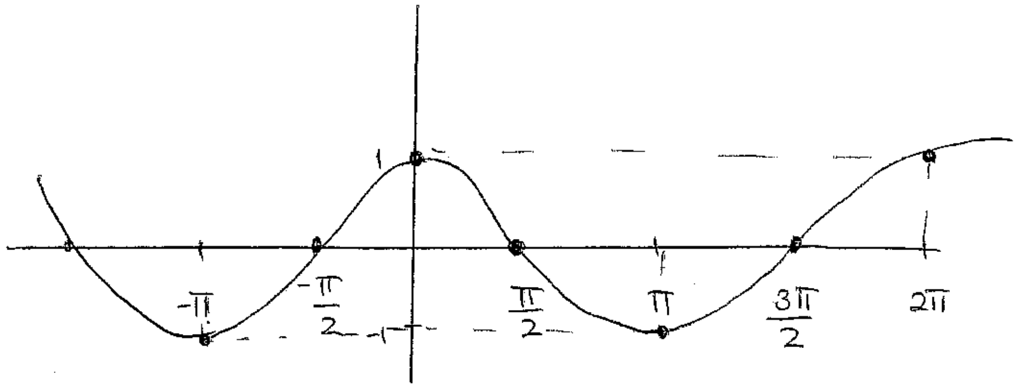
$$\cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right) = -\frac{1}{2}$$



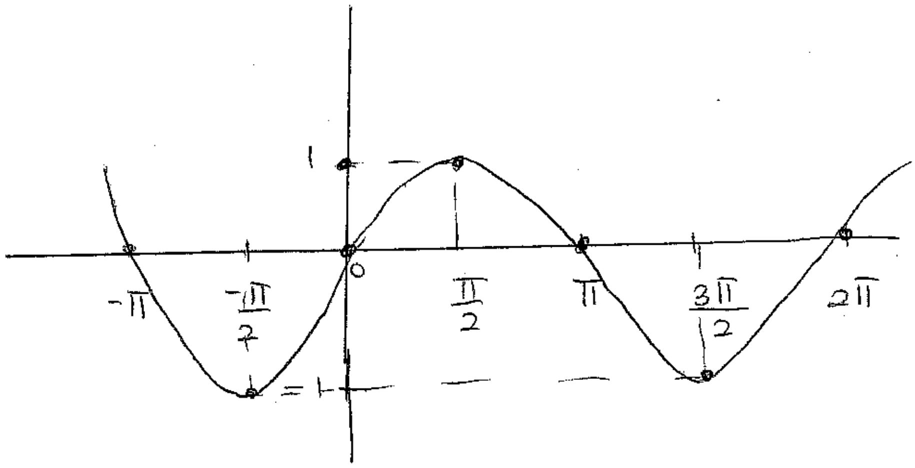
$$= \cos\left(\pi - \left(-\frac{\pi}{3}\right)\right) = -\cos\left(-\frac{\pi}{3}\right) = -\cos$$

$$= -\frac{1}{2}.$$

Graph of $\cos x$:



Graph of $\sin x$:



Additional Formulas For Cosine & Sine:

$$\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b$$

$$\sin(a \mp b) = \sin a \cos b \mp \cos a \sin b$$

Ex: Find $\cos\left(\frac{\pi}{12}\right) = \cos 16^\circ$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{3}\cdot\sqrt{2}}{4} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(2a) = \sin a \cos a + \cos a \sin a$$

$$\boxed{\sin 2a = 2\sin a \cos a.}$$

$$\cos(a+a) = \cos a \cos a - \sin a \sin a$$

$$\boxed{\cos 2a = \cos^2 a - \sin^2 a.}$$

Double - Angle Formulas.

since $\cos^2 a + \sin^2 a = 1$

OR $\boxed{\cos 2a = 1 - 2\sin^2 a}$

OR $\boxed{\cos 2a = 2\cos^2 a - 1}$

$$\boxed{\begin{aligned} \cos^2 a &= \frac{\cos 2a + 1}{2} \\ \sin^2 a &= \frac{1 - \cos 2a}{2} \end{aligned}}$$

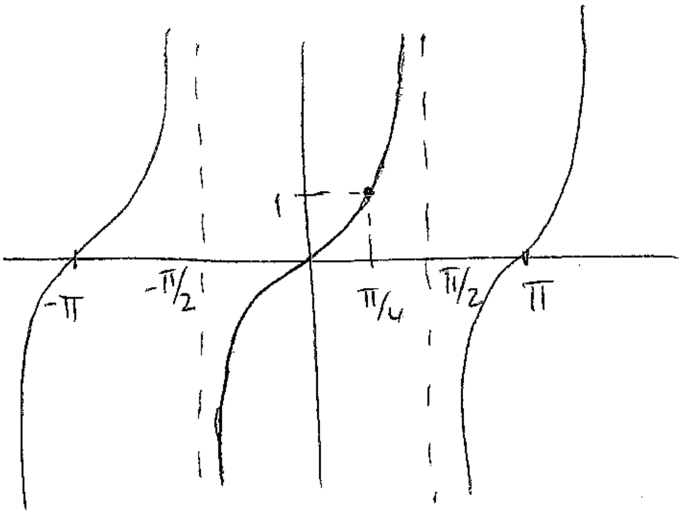
Half - angle formulas.

Other Trigonometric Functions

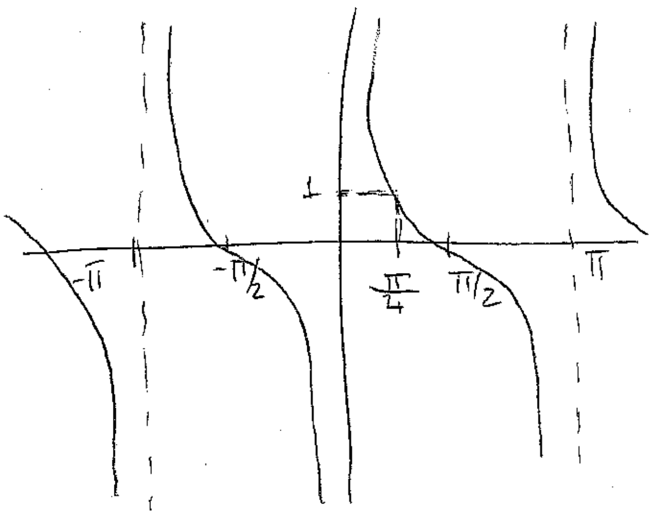
$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} \qquad \csc x = \frac{1}{\sin x}$$

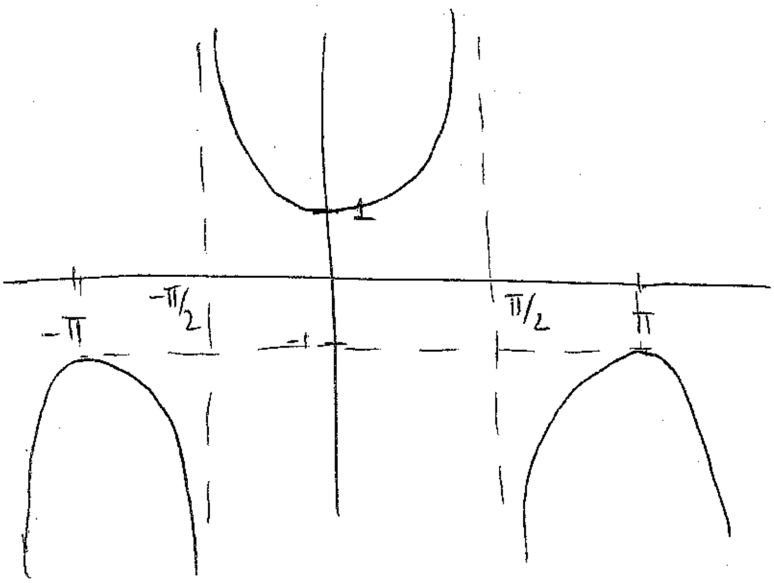
Graph of tan x



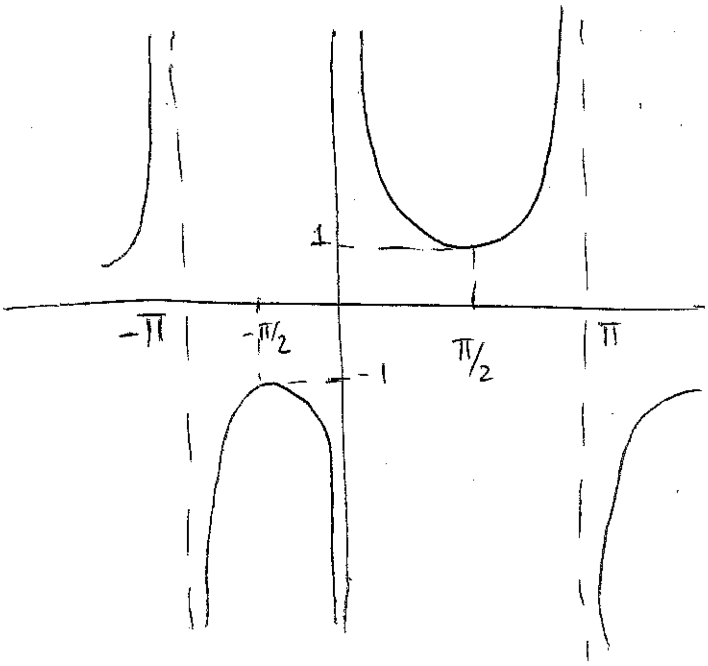
Graph of Cot x



Graph of Sec x



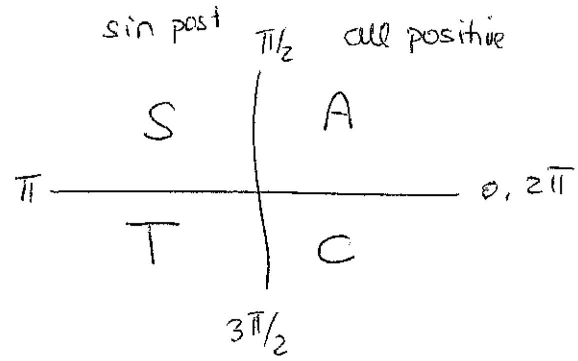
Graph of Csc x



Note: tan, Cot and Csc are odd functions.

Secant is even function

Remember:



Ex: Find $\sin\theta$, $\tan\theta$ where $\theta \in [\pi, \frac{3\pi}{2}]$

and $\cos\theta = -\frac{1}{3}$

Soln: $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9} \Rightarrow \sin\theta = \mp\sqrt{\frac{8}{9}}$$

but $\theta \in [\pi, \frac{3\pi}{2}] \Rightarrow \sin\theta$ is neg.

$$\Rightarrow \sin\theta = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3} \;//.$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-2\sqrt{2}}{\cancel{3}} \cdot \frac{-\cancel{3}}{1} = 2\sqrt{2} \;//$$

Remark:

* Secant and Cosecant are periodic with 2π

but tangent and cotangent " " " π

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{\sin x\cos\pi + \cos x\cancel{\sin\pi}^{0}}{\cos x\cos\pi - \sin x\cancel{\sin\pi}_{0}} = \frac{-\sin x}{-\cos x} = \tan x$$

So, $\boxed{\tan(x+\pi) = \tan x.}$

Since, $\cos^2 x + \sin^2 x = 1.$ $(*)$

divide $(*)$ by $\cos^2 x$ $\Rightarrow$ $\boxed{1+\tan^2 x = \sec^2 x.}$

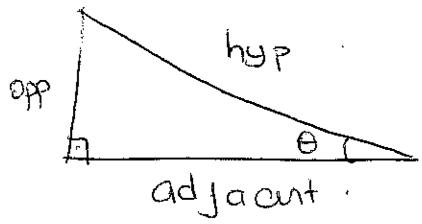
divide $(*)$ by $\sin^2 x$ $\Rightarrow$ $\boxed{\cot^2 x + 1 = \csc^2 x}$

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a\cos b + \cos a\sin b}{\cos a\cos b - \sin a\sin b}$$

$$= \frac{\frac{1}{\cos a\cos b}(\sin a\cos b + \cos a\sin b)}{\frac{1}{\cos a\cos b}(\cos a\cos b - \sin a\sin b)}$$

$$\boxed{\tan(a\mp b) = \frac{\tan a\mp\tan b}{1\pm\tan a\tan b}}$$

Trigonometric Review

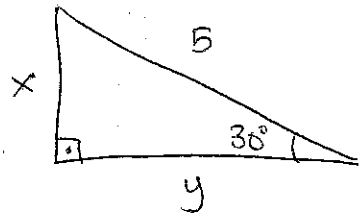


$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{opp}{adj}$$

Ex:

Find x and y.

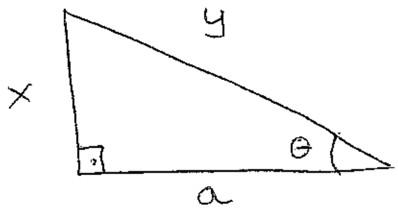


Soln:

$$\sin 30^\circ = \frac{x}{5} = \frac{1}{2} \Rightarrow x = \frac{5}{2} \;//$$

$$\cos 30^\circ = \frac{y}{5} = \frac{\sqrt{3}}{2} \Rightarrow y = \frac{5\sqrt{3}}{2} \;//$$

Ex:



Express x and y in terms of a and angle θ.

Soln:

$$\tan\theta = \frac{x}{a} \Rightarrow x = a\tan\theta$$

$$\cos\theta = \frac{a}{y} \Rightarrow y = \frac{a}{\cos\theta} = a\sec\theta.$$

CHAPTER 1 Limits & Continuity :

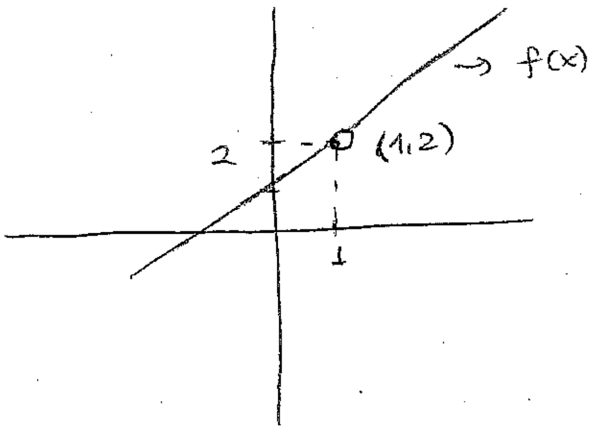
to talk about rate of change, tangent lines and areas bounded by curves → we need process of finding limits.

1.2 Limits of Functions

Ex 1 Describe the behaviour of

$$f(x) = \frac{x^2-1}{x-1} \quad \text{near} \quad x=1$$

Soln:



$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1$$

f is defined for all x ~~except~~ $x \neq 1$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Defn 1 Informal Defn of Limit.

If $f(x)$ is defined for all x near a, except at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a,

then, f approaches the limit L as x approaches a,

and $\lim_{x\to a} f(x) = L$

Note: Sometimes

$$\lim_{x \to a} f(x) = f(a)$$

This is the case when

- $f(x)$ defined in an open interval containing $x=a$
- graph of f passes unbroken through $(a, f(a))$

Ex: Find,

(a) $\lim_{x \to a} x = a$

(b) $\lim_{x \to a} c = c$ $\quad$ c: constant.

Ex: Find,

(a) $$\lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$$

undefined at $x = -2$.

$$= \lim_{x \to -2} \frac{(x-1)\cancel{(x+2)}}{\cancel{(x+2)}(x+3)} = \lim_{x \to -2} \frac{x-1}{x+3} = \frac{-3}{1} = -3$$

(b) $$\lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}$$ undefined at $x=a$.

$$= \lim_{x \to a} \frac{\frac{a-x}{xa}}{x-a} = \lim_{x \to a} \frac{\frac{-(x-a)}{xa}}{x-a} = \lim_{x \to a} -\frac{1}{xa} = -\frac{1}{a^2}$$

(c.) $\lim_{x \to 4} \dfrac{\sqrt{x}-2}{x^2-16} =$ undefined at $x=4$.

$$= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{\cancel{x-4}}{(\cancel{x-4})(x+4)(\sqrt{x}+2)}$$

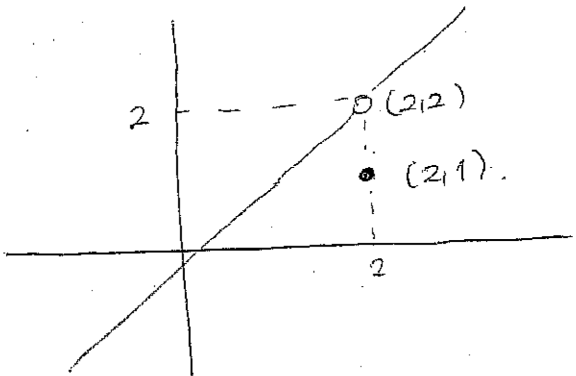
$$= \frac{1}{32} \quad //$$

<u>Note:</u> Existence of $\lim_{x \to a} f(x)$ depends only on the values of $f(x)$ for x near but not equal to a.

- Existence of $\lim_{x \to a} f(x)$ does not require $f(a)$ exist

  " depend on $f(a)$ even if $f(a)$ does exist.

<u>Ex:</u> Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



$\lim_{x \to 2} g(x) = \lim_{x \to 2} x = 2$

but $g(2) = 1$.

Defn 2 Left and Right Limits.

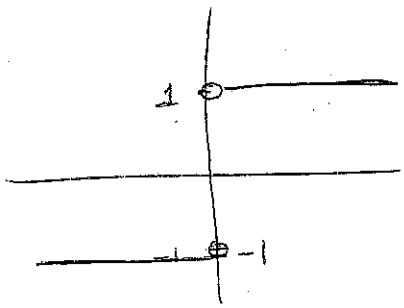
If $f(x)$ is defined on (a,b) and if $f(x)$ is close enough to number L when x is to the right of a and close enough to a, then $f(x)$ has right limit L at $x=a$.

$$\lim_{x\to a^+} f(x) = L$$

If $f(x)$ is defined on (c,a) and if $f(x)$ is close enough to number L when x is to the left of a and close enough to a, then $f(x)$ has left limit L at $x=a$

$$\lim_{x\to a^-} f(x) = L$$

Ex Let $f(x) = \mathrm{sgn}(x) = \dfrac{x}{|x|}$



$$\lim_{x\to 0^-} \mathrm{sgn}(x) = -1$$

$$\lim_{x\to 0^+} \mathrm{sgn}(x) = 1$$

Thm 1

$$\boxed{\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L}$$

Ex: Let $f(x) = \dfrac{|x-2|}{x^2+x-6}$

Find (a.) $\displaystyle\lim_{x\to 2^+} f(x)$

(b.) $\displaystyle\lim_{x\to 2^-} f(x)$

(c) $\displaystyle\lim_{x\to 2} f(x)$.

Soln: For $x > 2$ $\quad |x-2| = x-2$.

(a.) $$\lim_{x\to 2^+} \frac{x-2}{\underset{2\ \ 3}{x^2+x-6}} = \lim_{x\to 2^+} \frac{x-\cancel{2}}{(x+3)(x-\cancel{2})} = \frac{1}{5}$$

(b.) For $x < 2$ $\quad |x-2| = -(x-2) = \cancel{-x+2}$.

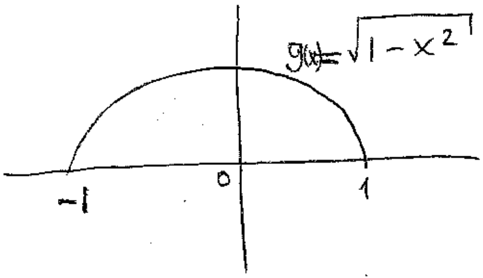
$$\lim_{x\to 2^-} \frac{|x-2|}{x^2+x-6} = \lim_{x\to 2^-} \frac{-(x-\cancel{2})}{(x+3)(x-\cancel{2})} = -\frac{1}{5}$$

(c.) Since $\displaystyle\lim_{x\to 2^-} f(x) = \frac{-1}{5} \neq \frac{1}{5} = \lim_{x\to 2^+} f(x)$.

$\Rightarrow$ $\displaystyle\lim_{x\to 2} f(x)$ does not exist.

Ex: What limits does $g(x) = \sqrt{1-x^2}$ have at $x = -1$ and $x = 1$

Soln:



$D_g = [-1, 1]$.

So, $\lim\limits_{x \to -1^+} g(x) = 0$ $\lim\limits_{x \to 1^-} g(x) = 0$

$\therefore$

$g(x)$ has no left limit or limit at $x = -1$

" no right " " $x = 1$

Thm 2 Limit Rules:

If $\lim\limits_{x \to a} f(x) = L$ $\lim\limits_{x \to a} g(x) = M$, k: constant.

1. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$

2. $\lim\limits_{x \to a} [f(x) - g(x)] = L - M$

3. $\lim\limits_{x \to a} f(x)\, g(x) = LM$

4. $\lim\limits_{x \to a} k f(x) = kL$

5. $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \qquad M \neq 0$$

6. $$\lim_{x\to a} [f(x)]^{m/n} = [\lim_{x\to a} f(x)]^{m/n} = L^{m/n}, \quad L>0 \text{ if } n = \text{ev}$$

$L \neq 0$ if $m<0$

7. if $f(x) \leq g(x)$ on interval containing a

$$\Rightarrow \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

$$\Rightarrow \qquad L \leq M.$$

<u>Note:</u> Rules (1) - (6) are also true for right & left limits.

<u>Ex:</u> Find,

(a) $$\lim_{x\to a} \frac{x^2+x+4}{x^3-2x^2+7} = \frac{a^2+a+4}{a^3-2a^2+7} \quad \text{where } a^3-2a^2+7 \neq$$

(b) $$\lim_{x\to 2} \sqrt{2x+1} = \lim_{x\to 2} (2x+1)^{1/2}$$

$$= \left[\lim_{x\to 2} (2x+1)\right]^{1/2} = 5^{1/2} = \sqrt{5}.$$

Defn: A <u>polynomial</u> in variable x, is a function

of the form,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$a_n, a_{n-1}, \ldots a_0$ are constants, $a_n \neq 0$ and $n =$ degree of $P(x)$.

The quotient $\dfrac{P(x)}{Q(x)}$ is called <u>rational function</u>

Ex

$2x^2 + 4 \rightarrow$ poly of deg $= 2$.

$5 \rightarrow$ " " deg $= 0$

$\dfrac{2x}{x+1} \rightarrow$ rational function.

Thm 3 Limits of Polynomials and Rational Functions.

1. $\lim_{x \to a} P(x) = P(a)$

2. $\lim_{x \to a} \dfrac{P(x)}{Q(x)} = \dfrac{P(a)}{Q(a)}$, $Q(a) \neq 0$.

The Squeeze Thm:

Suppose that $f(x) \le g(x) \le h(x)$ hold for all x in some open interval containing a, except possibly at $x=a$ itself.

Suppose also,

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then, $\lim_{x\to a} g(x) = L$ also.

Ex: Given,

$$3-x^2 \le u(x) \le 3+x^2 \quad \text{for all } x \ne 0$$

Find $\lim_{x\to 0} u(x)$

Soln:

$$\lim_{x\to 0}(3-x^2) = \lim_{x\to 0}(3+x^2) = 3$$

By squeeze thm $\lim_{x\to 0} u(x) = 3$.

Ex: Show that if $\lim_{x\to a} |f(x)| = 0 \Rightarrow \lim_{x\to a} f(x) = 0$.

Soln:

$$-|f(x)| \le f(x) \le |f(x)|$$

$$\lim_{x\to a} -|f(x)| = -1 \cdot \lim_{x\to a} |f(x)| = (-1)(0) = 0.$$

By squeeze thm, $\lim_{x\to a} f(x) = 0$.

Sec 1.3 Limits at Infinity and Infinite Limits.

Limits at Infinity:

Defn: If f is defined on (a, ∞) and if $f(x)$ is close enough to number L when x is large enough then we say that, $f(x)$ approaches the limit L as x approaches infinity

$$\lim_{x \to \infty} f(x) = L.$$

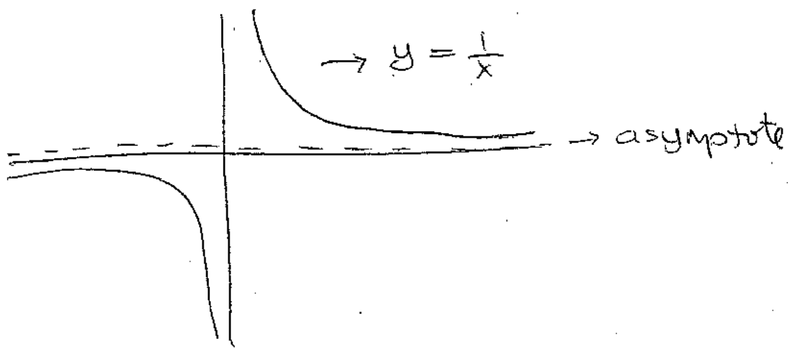
If f is defined on $(-\infty, b)$ and if $f(x)$ is close enough to number M when x is large enough in abs. value

$\Rightarrow$ $f(x)$ approaches the limit M as $x \to -\infty$

$$\lim_{x \to -\infty} f(x) = M$$

Ex 1 $\lim_{x \to \infty} \frac{1}{x} = ?$

$\lim_{x \to -\infty} \frac{1}{x} = ?$



$\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$.

Ex 2

Let $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

(a.) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2(1+\frac{1}{x^2})}}$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{(1+\frac{1}{x^2})}} = \lim_{x\to\infty} \frac{x}{|x|} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$$

$$sgn(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 . \end{cases}$$

$$= \lim_{x\to\infty} \frac{x}{|x|} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

(b.) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x}{|x|} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} = -1 .$

Limits at Infinity For Rational Functions :

In this case divide the numerator and denominator by the highest power of x appearing in the denominator

Ex : 3 $\lim\limits_{x\to\mp\infty} \dfrac{2x^2-x+3}{3x^2+5}$ (Numerator and denominator have same degree)

$$= \lim_{x\to\mp\infty} \frac{\frac{2x^2}{x^2} - \frac{x}{x^2} + \frac{3}{x^2}}{3x^2/x^2 + 5/x^2} = \lim_{x\to\mp\infty} \frac{2-(\frac{1}{x})+(\frac{3}{x^2})}{3+(5/x^2)} = \frac{2}{3}$$

<u>Ex: 4</u> $\lim_{x \to \mp\infty} \frac{5x+2}{2x^3-1}$ deg of num < deg of denom

$$\Rightarrow \lim_{x \to \mp\infty} \frac{(5x/x^3)+(2/x^3)}{(2x^3/x^3)-(1/x^3)} = \lim_{x \to \mp\infty} \frac{(5/x^2)+(2/x^3)}{2-(1/x^3)}$$

$$= \frac{0}{2} = 0$$

<u>Summary:</u>

Let $P_m(x) = a_m x^m + \cdots + a_0 \longrightarrow \deg m$

$Q_n(x) = b_n x^n + \cdots + b_0 \longrightarrow \deg n.$

$\Rightarrow$ 1) $\lim_{x \to \pm\infty} \frac{P_m(x)}{Q_n(x)} = 0$ if $m < n$

2.) $\lim_{x \to \mp\infty} \frac{P_m(x)}{Q_n(x)} = \frac{a_m}{b_n}$ if $m = n$.

3.) $\lim_{x \to \mp\infty} \frac{P_m(x)}{Q_n(x)} =$ does not exist if $m > n$.

<u>Ex:</u> Find $\lim_{x \to \infty} (\sqrt{x^2+x} - x)$

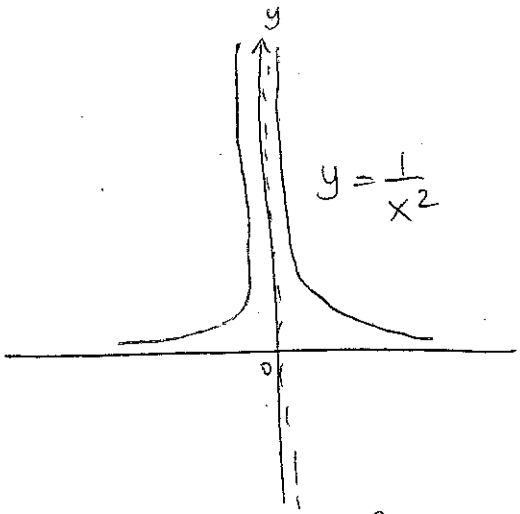
<u>Soln:</u> $\lim_{x \to \infty} (\sqrt{x^2+x} - x) = \lim_{x \to \infty} \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{(\sqrt{x^2+x} + x)}$

$$= \lim_{x \to \infty} \frac{\cancel{x^2}+x-\cancel{x^2}}{\sqrt{x^2(1+\frac{1}{x})}+x} = \lim_{x \to \infty} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+\frac{1}{x}} + \cancel{1})} =$$

$\sqrt{x^2} = x$ because $x > 0$

Infinite Limits

Ex: Find $\lim_{x \to 0} \frac{1}{x^2}$



$\lim_{x \to 0} \frac{1}{x^2} = \infty.$ → means limit does not exis

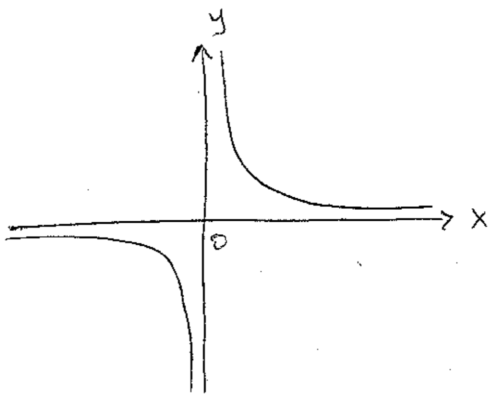
$\frac{1}{x^2}$ becomes very large near $x = 0$.

Note: if a curve approaches a straight line as it goes very far away from the origin, that line is called asymptote of curve.

Ex: Find,

a.) $\lim_{x \to 0^+} \frac{1}{x}$

b.) $\lim_{x \to 0^-} \frac{1}{x}$



(a.) $\lim_{x \to 0^+} \frac{1}{x} = \infty$ limit does not exist.

(b) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Ex:

(a) $\lim\limits_{x\to\infty} (3x^3 - x^2 + 2) = \lim\limits_{x\to\infty} 3x^3\left(1 - \frac{1}{3x} + \frac{2}{3x^3}\right) = \infty$

(b) $\lim\limits_{x\to\infty} (x^4 - 5x^3 - x) = \lim\limits_{x\to\infty} x^4\left(1 - \frac{5}{x} - \frac{1}{x^3}\right) = \infty$

(c.) $\lim\limits_{x\to-\infty} (3x^3 - x^2 + 2) = \lim\limits_{x\to-\infty} 3x^3\left(1 - \frac{1}{3x} + \frac{2}{3x^3}\right) = -\infty$

(d.) $\lim\limits_{x\to-\infty} (x^4 - 5x^3 - x) = \lim\limits_{x\to-\infty} x^4\left(1 - \frac{5}{x} - \frac{1}{x^3}\right) = \infty$

Ex: Evaluate,

$$\lim_{x\to\infty} \frac{x^3+1}{x^2+1} = \lim_{x\to\infty} \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x^2}} =$$

$$\lim_{x\to\infty} \frac{x + \frac{1}{x^2}}{1} = \infty$$

Ex: Evaluate.

(a.) $\lim\limits_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim\limits_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{x-2}{x+2} = 0$

(b.) $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \lim\limits_{x\to 2} \frac{(x-2)}{(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

(c.) $\lim\limits_{x\to 2^+} \frac{x-3}{x^2-4} = \lim\limits_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

for $x > 2$.
value of funct is neg
when x near 2

$(0.1)(4)$ - - $\rightarrow \frac{1}{2}$

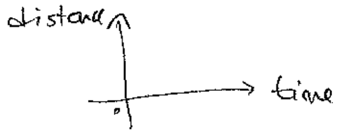
(d.) $\lim_{x \to 2^-} \frac{x-3}{x^2-4} = \lim_{x \to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ the values of f is post for $x<2$ near $x=$

S

(e) $\lim_{x \to 2} \frac{x-3}{x^2-4}$ does not exist. $\lim_{x \to 2^-} \frac{x-3}{x^2-4} \neq \lim_{x \to 2^+} \frac{(x-3}{x^2-4}$

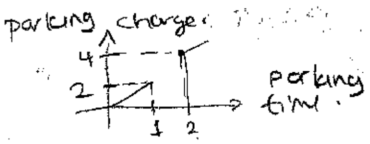
(f.) $\lim_{x \to 2} \frac{2-x}{(x-2)^3} = \lim_{x \to 2} \frac{-(x-2)}{(x-2)^3} = \frac{-1}{(x-2)^2} = -\infty$

1.4 Continuity :



car drive in highway its distance from its starting point depends on time in cont. way

suppose car parked in a parking lot, $\rightarrow$ charges \$2 per hour or portion.



discontinuous funct.

Defn: If domain of f is an interval or union of separat intervals then a point P in the domain of f is an interior point of the domain if P belongs to some open interval contained in the domain.

Otherwise P is called endpoint.

Ex: $f(x) = \sqrt{4-x^2}$ $\quad d_f = [-2,2]$.

f has interior points in $(-2,2)$.

-2 is left endpoint, 2 is right endpoint.

Defn: Continuity at an Interior Point

A function f is continuous at an interior point c of its domain if

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

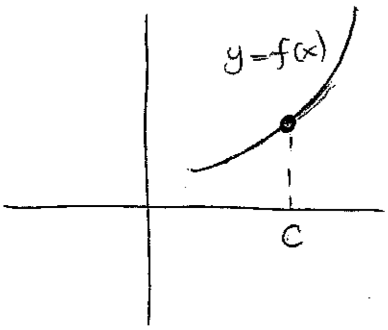
if $\lim\limits_{x \to c} f(x)$ does not exist

OR if $\lim\limits_{x \to c} f(x) \neq f(c)$

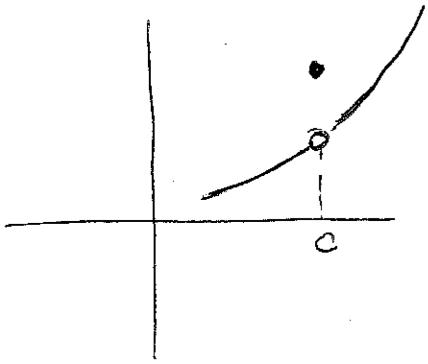
$\Big\} \Rightarrow$ f is <u>discontinuous</u> <u>at c</u>.

<u>Ex</u>

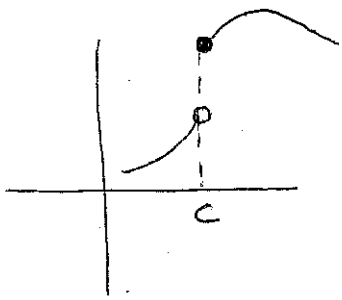
left, right limit are not eq



f is continuous at c



$\lim\limits_{x \to c} f(x) \neq f(c)$



$\lim\limits_{x \to c} f(x)$ does not exist

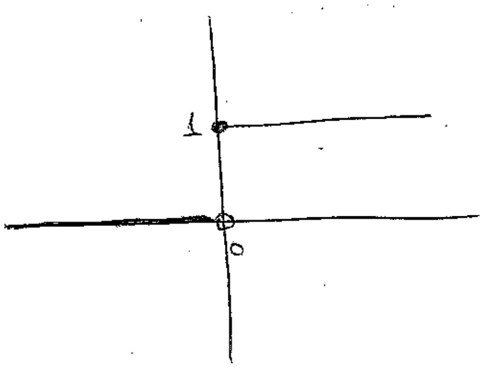
<u>Defn:</u> Right & Left Continuity:

f is <u>right continuous</u> at c if $\boxed{\lim\limits_{x \to c^+} f(x) = f(c)}$

f is <u>left continuous</u> at c if $\boxed{\lim\limits_{x \to c^-} f(x) = f(c)}$

<u>Ex:</u>

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



$\lim\limits_{x \to 0^+} H(x) = H(0) = 1$ right continuous.

$\lim\limits_{x \to 0^-} H(x) = 0 \neq H(0) = 1$ not left continuous.

Thm: $\boxed{f \text{ is continuous at } c \iff \text{it is both left and right continuous at } c}$

$$\text{i.e.}\quad \lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = f(c)$$

Defn: Continuity at an endpoint

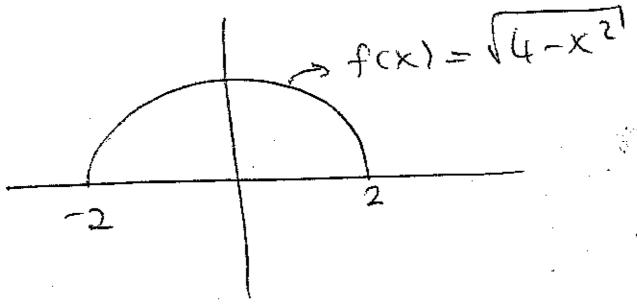
f is continuous at a left endpoint c of its domain if it is right continuous there.

f is continuous at a right endpoint c of its domain if it is left continuous there.

Ex:

$$f(x) = \sqrt{4-x^2}$$

$$d_f = [-2,2].$$



$$\lim_{x \to 2-} f(x) = f(2) = 0.$$

$\therefore$ continuous at right end point

$$\lim_{x \to -2-} f(x) = f(-2) = 0$$

$\therefore$ continuous at left end point

Also, for $-2 < c < 2$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{4-x^2} = \sqrt{4-c^2} = f(c)$$

$\therefore$ it is continuous at every interior point of its domain

Continuity on an Interval :

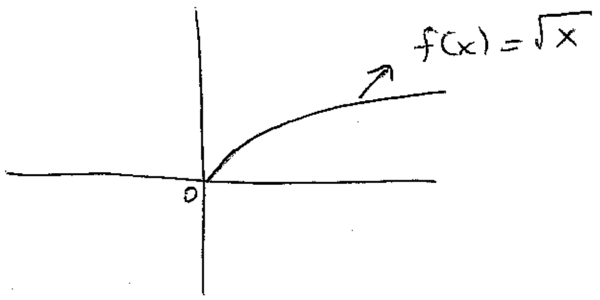
f is continuous on the interval I if it is continuous at each point of I.

f is continuous function if f is continuous at every point of its domain.

Ex: $f(x) = \sqrt{x}$

$D_f = [0, \infty)$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = 0 = f(0)$



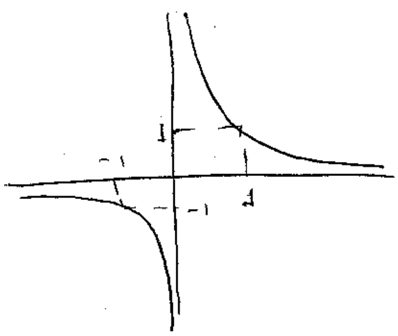
f is right continuous at endpoint 0.

$\lim_{x \to c} \sqrt{x} = \sqrt{c} = f(c) \quad \forall\, c > 0.$

f is continuous for $\forall\, c > 0$.

So, f is a continuous function.

Ex: $g(x) = \frac{1}{x}$ is continuous function on its domain.



we can't say it is discontinuous at $x = 0$ because $x = 0 \notin D_g$.

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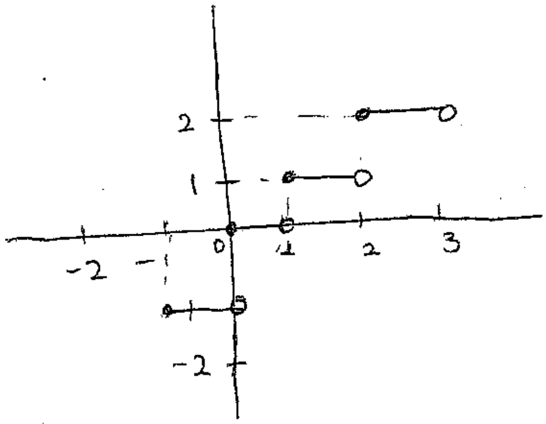
Ex:

$f(x) = \lfloor x \rfloor$ greatest integer function.

continuous on $\forall$ $[n, n+1)$ n: integer.



$\lim_{x \to n^+} \lfloor x \rfloor = n = \lfloor n \rfloor$

So it is right continuous.

$\lim_{x \to n^-} \lfloor x \rfloor = n-1 \neq n = \lfloor n \rfloor$

Not left continuous

it is discontinuous at ~~each~~ integers

Ex:

$\lim_{x \to 1^+} \lfloor x \rfloor = 1 = \lfloor 1 \rfloor$

$\lim_{x \to 1^-} \lfloor x \rfloor = 0 \neq \lfloor 1 \rfloor = f(1) = 1$

Examples

$$\left.\begin{array}{l} \text{1.) all polynomials} \\ \text{2.) " rational functions} \\ \text{3.) " rational powers } x^{m/n} = \sqrt[n]{x^m} \\ \text{4.) sine, cosine, tan, sec, csc, cot} \end{array}\right\} \text{continuous wherever they are define}$$

5.) abs value funct.

Thm 6 Let f, g are defined on an interval containing c and they are continuous at c.

$\Rightarrow$ following functions are also continuous at c.

1) $f-g$, $f+g$

2.) fg

3.) kf , k: any number

4.) $\frac{f}{g}$, $g(c) \neq 0$

5.) $(f(x))^{1/n}$, $f(c) > 0$ if n: even.

$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$
$$= f(c) + g(c)$$
$\Rightarrow f+g$ is cont.

Thm 7 Composites of continuous functions are continuous.

If $f(g(x))$ is defined on an interval c, if f is continuous at L and $\lim_{x \to c} g(x) = L$

$$\Rightarrow \lim_{x \to c} f(g(x)) = f(L) = f(\lim_{x \to c} g(x))$$

In particular,

if g is cont at c, so $g(c) = L$

$\Rightarrow f \circ g$ is continuous at c,

$$\lim_{x \to c} f(g(x)) = f(g(c))$$

Ex: Following functions are cont.

a.) $4x^2 - x \rightarrow$ it is polynomial

b.) $\dfrac{x+2}{x^2-4} \rightarrow$ all rationals are cont.

c.) $|x^2-1| \rightarrow \underbrace{f(x) = |x|}_{\text{cont.}} \quad \underbrace{g(x) = x^2-1}_{\text{cont}} \quad f \circ g = |x^2-1|$

d.) $\sqrt{x^2-2x-5} \rightarrow \underbrace{f(x) = \sqrt{x}}_{\text{cont}} \quad \underbrace{g(x) = x^2-2x-5}_{\text{cont}}. \quad f \circ g = \sqrt{x^2-2x-5}$

e.) $\dfrac{\overbrace{|x|}^{\text{cont.}}}{\underbrace{\sqrt{|x+2|}}_{\text{cont}}} \rightarrow$ cont.

Continuous Extensions and Removable Discontinuities

If $f(c)$ is not defined,

but $\lim\limits_{x \to c} f(x) = L$ exists

$\Rightarrow$ define,
$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in domain of } f \\ L & \text{if } x = c. \end{cases}$$

$F(x)$ is cont. at $x = c$

$F(x)$ is called <u>continuous extension</u> of $f(x)$ to $x = c$.

Note: For rational functions f, extensions are found by cancelling common factors.

Ex: Show that $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ has a continuous extension to $x = 1$, and find that extension.

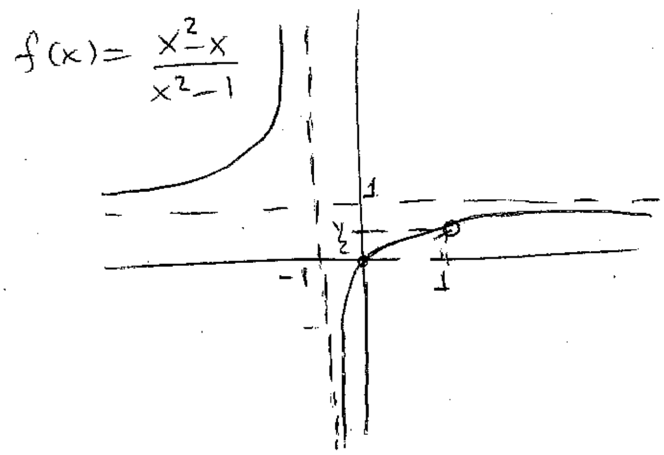
Soln:

$f(1)$ is not defined.

$$f(x) = \frac{x(x-1)}{(x-1)(x+1)} = \frac{x}{x+1} \quad \text{if } x \neq 1$$

$$\lim_{x \to 1} f(x) = \frac{1}{1+1} = \frac{1}{2}$$

$$F(x) = \begin{cases} \dfrac{x}{x+1} & \text{for } x \neq 1 \\ \dfrac{1}{2} & x = 1 \end{cases}$$



$F(x)$ has same graph as $f(x)$ with no hole at $(1, \frac{1}{2})$

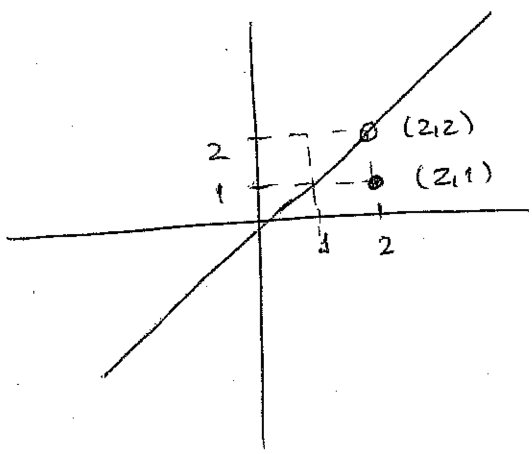
Defn: Removable Discontinuity

If f that is discontinuous at a point a can be redefine at that single point so that it becomes continuous there

$\Rightarrow$ f has removable discontinuity at a.

Ex:

$$g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$



has removable discontinuity at $x=2$.

to remove,

define, $g(2) = 2$

Continuous Functions on Closed, Finite Intervals:

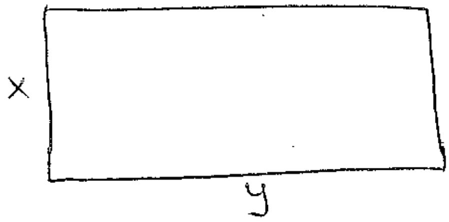
Thm 8 The Max-Min Thm:

If $f(x)$ is continuous on the closed, finite interval $[a,b]$, then $\exists$ x_1 and x_2 in $[a,b]$ s.t for $\forall x \in [a,b]$.

$$\underbrace{m = f(x_1)}_{\substack{\downarrow \\ \text{abs. min value of } f \\ \text{at } x_1}} \leq f(x) \leq \underbrace{f(x_2) = M}_{\substack{\downarrow \\ \text{abs. max vale of } f \\ \text{at } x_2}}$$

Ex: what is the largest possible area of a rectangular field that can be enclosed by 200 m of fencing?

Soln:



perimeter $= 2x + 2y = 200$

$\Rightarrow x + y = 100$

$y = 100 - x$

Area $= xy \ m^2$

$A(x) = x(100 - x) = 100x - x^2$

$(x-50)^2 = x^2 - 100x + 2500$

$A(x) = 2500 - (x-50)^2 \quad \Rightarrow$ vertex point $(50, 2500)$.

max value of $A(x)$ is 2500 when $x = 50$, $y = 50$.

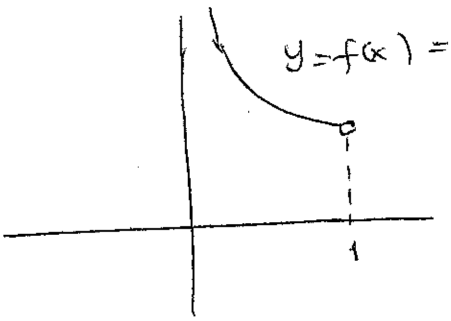
Note: Thm 8 $\Rightarrow$ If f is continuous on closed, finite interval $\Rightarrow$ f is bounded

ie, f can't take arbit large post or neg values.

ie, $-K \le f(x) \le K \qquad |f(x)| \le K$.

NO

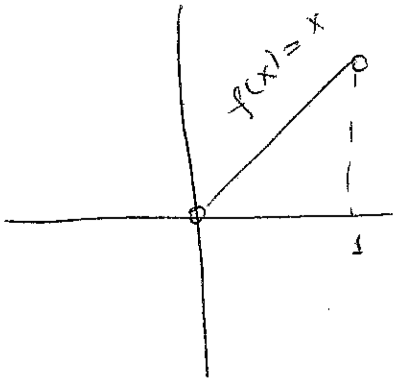
Finding Maxima & Minima Graphically:



continuous on $(0,1)$

neither max nor min

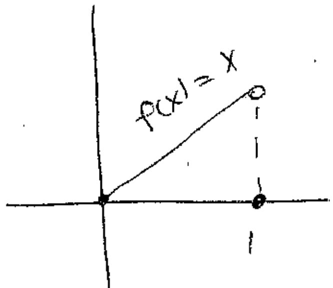
not bounded.



continuous on $(0,1)$

neither max nor min.

bounded.



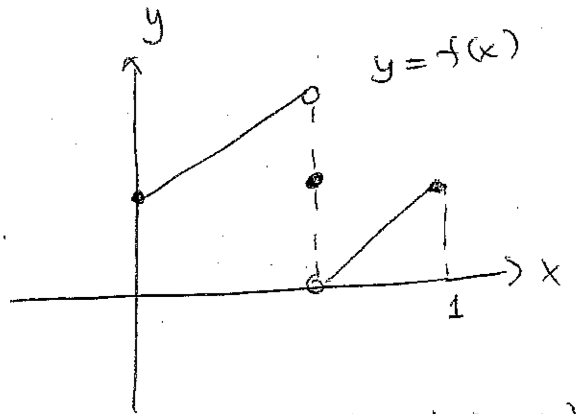
f defined on

~~Dg~~ = $[0,1]$.

discont at $x=$

minval $= 0$

no max val



f is discont at interior pts.

it is bounded, $[0,1]$

neither max, nor min.

Thm 9: The Intermediate-Value Thm:

If $f(x)$ is continuous on $[a,b]$ and if $f(a) \leq s \leq f(b)$

$\Rightarrow \exists \; c \in [a,b]$ s.t $f(c) = s$.

Ex: Determine the intervals on which

$f(x) = x^3 - 4x$ is positive and negative.

Soln:

$$f(x) = x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$$

$f(x) = 0$ when $x = 0$, $x = 2$, $x = -2$.

f is cont on $\mathbb{R}$ $\Rightarrow$ it has constant sign on each interval.

$$(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$$

ie if a, b are points in $(0,2)$ s.t $f(a) < 0$
$f(b) > 0$

then by intermediate val thm $\exists$ c s.t $a < c < b$ $f(c) = 0$

but f has no zero in $(0,2)$.

- Take a point on each interval:

$x = -3$ $f(-3) = -15 < 0 \Rightarrow f(x)$ is negative on $(-\infty, -2)$

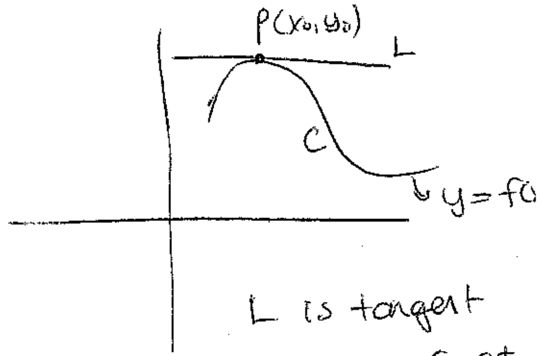
$x = -1$ on $(-2, 0)$ $f(-1) = 3 > 0 \Rightarrow f$ is $+$ on $(-2, 0)$

$x = 1$ on $(0, 2)$ $f(1) = -3 < 0 \Rightarrow f$ is $-$ on $(0, 2)$

$x = 3$ on $(2, \infty)$ $f(3) = 15 > 0 \Rightarrow f$ is $+$ on $(2, \infty)$.

Chapter 2 DIFFERENTIATION

2-1 Tangent Lines & Their Slopes :



L is tangent to curve C at point P.

Defn 1 Non-vertical Tangent Lines

Suppose, f is continuous at $x = x_0$

and $\boxed{\lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = m}$ exists

Newton quotient.

$\Rightarrow$ the straight line having slope m and passing through $P = (x_0, f(x_0))$ is called tangent line (tangent) to $y = f(x)$ at P.

Eqn of tangent is $\boxed{y = m(x - x_0) + y_0.}$

Ex Find an eqn of the tangent line to curve $y = x^2$ at point $(1,1)$.

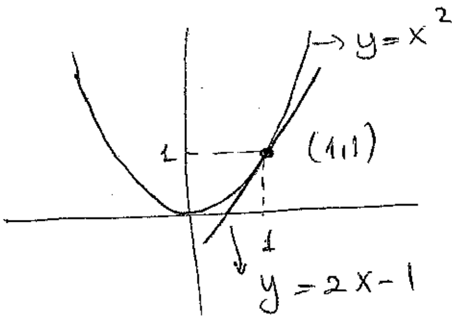
Soln: $f(x) = x^2$, $x_0 = 1$ $y_0 = 1 = f(1)$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = 2.$$

Eqn of tangent:

$$y = 2(x-1)+1 \Rightarrow y = 2x-1$$



<u>Note:</u> Graph of continuous function can have a <u>vertical tangent line</u>

<u>Defn 2</u> Vertical tangents:

If f is continuous at $P=(x_0, y_0)$, $y_0 = f(x_0)$ and

if either

$$\boxed{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \infty \quad \text{or} \quad \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = -\infty}$$

$\Rightarrow$ ~~the~~ the line $x = x_0$ is <u>vertical tangent</u> to $y = f(x)$ at P.

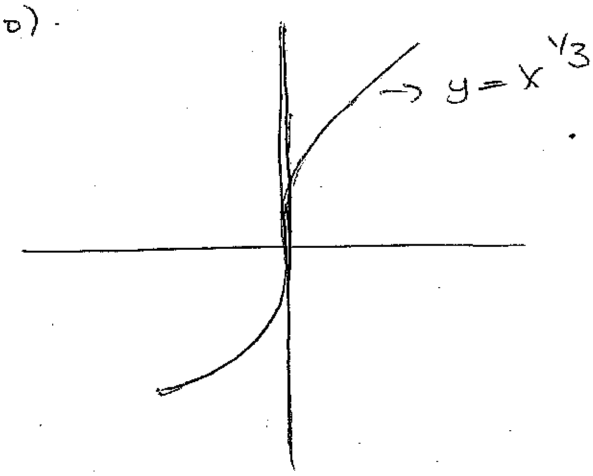
If <u>limit does not exist</u> in any other way than by being ∞ or $-\infty$ $\Rightarrow$ $y=f(x)$ has <u>no tangent line</u> at P

2

Ex: Find eqn of tangent line to $f(x) = \sqrt[3]{x} = x^{1/3}$ at $P(0,0)$.

Soln: $\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^{1/3}}{h} = \lim\limits_{h\to 0} \dfrac{1}{h^{2/3}} = \infty$

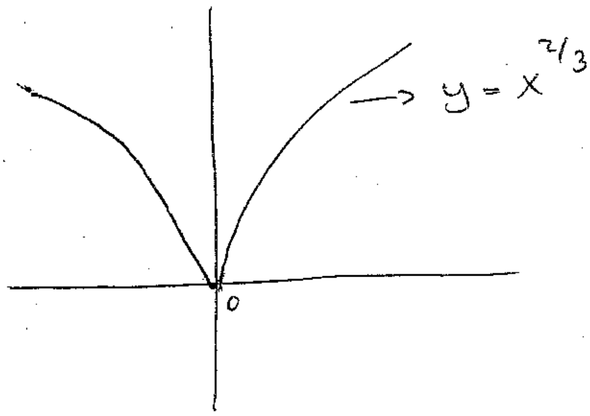
$\Rightarrow$ $x = 0$ is vertical tangent to to $f(x) = \sqrt[3]{x}$ at $(0,0)$.



vertical line $x=0$ or y-axis tangent to $y = x^{1/3}$ at $(0,0)$

?

Ex: $f(x) = x^{2/3}$



It does not have tangent at the origin because it is not smooth there.

$\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^{2/3}}{h} =$

$\lim\limits_{h\to 0} \dfrac{1}{h^{1/3}}$ has no limit.

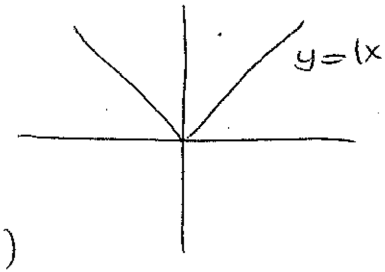
$\lim\limits_{h\to 0^+} \dfrac{1}{h^{1/3}} = \infty$

$\lim\limits_{h\to 0^-} \dfrac{1}{h^{1/3}} = -\infty$

Ex: Does the graph of $y = |x|$ have a tangent line at $x = 0$?

Soln: $\dfrac{f(x_0+h) - f(x_0)}{h} = \dfrac{f(0+h) - f(0)}{h} = \dfrac{|h|}{h} = \operatorname{Sgn} h = \begin{cases} 1 & \text{if } h \\ -1 & \text{if } h \end{cases}$

$\lim\limits_{h \to 0^+} \operatorname{Sgn} h = 1 \qquad \lim\limits_{h \to 0^-} \operatorname{Sgn} h = -1$



So $\operatorname{Sgn} h$ has no limit as $h \to 0$.

$\therefore$ $y = |x|$ has no tangent line at $(0,0)$

Note: $y = x^{2/3}$ and $y = |x|$ have tangent lines everywher except at the origin.

Defn 3 The slope of a curve

The slope of a curve C at a point P is the slope of the tangent line to C at P if such a tangent line exists.

In particular,

slope of $y = f(x)$ at x_0 is,

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Ex: Find the slope of $y = \dfrac{x}{3x+2}$ at the point $x = -2$

$-6+3h+2$

$-4+3h.$

Soln:

$$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \frac{\dfrac{-2+h}{3(-2+h)+2} - \dfrac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{4}+2h+\cancel{4}-3h}{2h(-4+3h)} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(-8+6h)} = \frac{1}{8}$$

Normals:

Defn If curve C has a tangent line L at point P, then the straight line N through P $\perp$ L is called the normal to C at P

* If L is horizontal $\Rightarrow$ N is vertical
* If L is vertical $\Rightarrow$ N is horizontal..
* if L is neither horizontal nor vertical $\Rightarrow$ ~~slp~~

$$\boxed{\text{slope of the normal} = \frac{-1}{\text{slope of the tangent}}}$$

Ex: Find an eqn of normal to $y = x^2$ at $(1,1)$

Soln: By previous ex the tangent line L to $y = x^2$ is neither horizontal, nor vertical and its slope $= 2$.

$\Rightarrow$ Slope of normal $= -\frac{1}{2}$.

$$y - 1 = -\frac{1}{2}(x-1) = -\frac{1}{2}x + \frac{1}{2}$$

$\boxed{y = -\frac{1}{2}x + \frac{3}{2}} \rightarrow$ eqn of normal.

Ex: Find eqns of the straight lines that are tangent and normal to the curve $y = \sqrt{x}$ at $(4,2)$.

Soln:

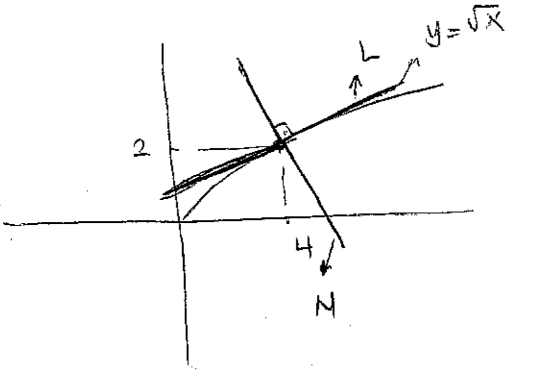
$$m = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{\cancel{4} + \cancel{h} - \cancel{4}}{\cancel{h}(\sqrt{4+h} + 2)} = \frac{1}{4}$$

$m = \frac{1}{4}$ slope of tangent line.

Eqn of tangent line:

$$y - 2 = \frac{1}{4}(x-4) = \frac{1}{4}x - 1 \Rightarrow \boxed{y = \frac{1}{4}x + 1}$$



slope of normal line $= -4$.

Eqn of normal line:

$$y - 2 = -4(x-4) = -4x + 16$$

$$\boxed{y = -4x + 18}$$

Sec 2.2 The Derivative

Defn: The derivative of a function f is,

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

at all points x for which the limit exists.

If $f'(x)$ exists, $\Rightarrow$ f is differentiable at x

Eqn of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)} \qquad y - f(x_0) = f'(x_0)(x - x_0)$$

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$D(f') =$ contains points in $D(f)$ at which f is differentiable

Defn: Values of x in $D(f)$ where f is not differentiable are called singular points of f.

Defn: If f is defined on a closed interval $[a,b]$ then,

f has right derivative at $x = a$

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{exists}$$

f has left derivative at $x = b$

$$f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{exists}$$

f is <u>differentiable</u> on $[a,b]$ if $f'(x)$ exists for all x in (a,b) and $f_+'(a)$ and $f_-'(b)$ both exist.

<u>Ex:</u> Show that

$$\text{if } f(x) = ax + b \Rightarrow f'(x) = a.$$

<u>Soln:</u>

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a(x+h) + b - (ax+b)}{h}$$

$$= \lim_{h\to 0} \frac{a\cancel{h}}{\cancel{h}} = a$$

$$\boxed{\text{If } f(x) = c \quad c: \text{constant} \Rightarrow f'(x) = 0}$$

<u>Ex</u> Use the defn of the derivative to calculate the derivative of

a-) $f(x) = x^2 - 3x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h} = \lim_{h\to 0} \frac{\cancel{h}(+2x + h - 3)}{\cancel{h}}$$

$$= +2x - 3 \text{ //}$$

(b.) $f(x) = \frac{1}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{x^2 h (x^2+2xh+h^2)} = \lim_{h\to 0} \frac{\cancel{h}(-2x-\cancel{h}^{0})}{\cancel{h}\, x^2 (x^2+2\cancel{hx}+\cancel{h^2})} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

(c.) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \quad //$$

So,

$$\boxed{\text{if } f(x) = x^r \Rightarrow f'(x) = r x^{r-1}}$$

Differentiating powers.

Ex: $f(x) = x^{2/3} \Rightarrow f'(x) = \frac{2}{3} x^{-1/3}$

Ex: Show if

$$\boxed{f(x) = |x| \Rightarrow f'(x) = \frac{x}{|x|} = \operatorname{Sgn} x}$$

Soln: $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

So $f'(x) = \operatorname{Sgn} x = \frac{x}{|x|}$

If $y = f(x)$ then

notation: derivative of function wrt x

$$y' = D_x y = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = Df(x).$$

The value of the derivative of a function at a particular number x_0 is.

$$D_x y \Big|_{x=x_0} = y' \Big|_{x=x_0} = \frac{dy}{dx}\Big|_{x=x_0} = \frac{d}{dx} f(x)\Big|_{x=x_0} = f'(x_0) = Df(x_0)$$

Ex: Use defn of derivative to find,

$f'(0)$ if $f(x) = \frac{2-x}{2+x}$

Soln: $f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{2-h}{2+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2} - h - \cancel{2} - h}{h(2+h)} = \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(2+h)} = \frac{-2}{2} = -1$$

2.3 Differentiation Rules

Thm1 If f is differentiable at x $\Rightarrow$ f is continuous at x.

Thm2 If f, g are differentiable at x

$\Rightarrow$ $f+g$, $f-g$, cf are also differentiable at x. c: constant

$$(f+g)'(x) = f'(x) + g'(x)$$

$$(f-g)'(x) = f'(x) - g'(x)$$

$$(cf)'(x) = c f'(x).$$

In general,

$$(f_1+f_2+\cdots+f_n)' = f_1' + f_2' + \cdots + f_n'$$

Ex: Calculate the derivative of

a.) $f(x) = 3x^2 - 5x - 7$

$$f'(x) = 3(2x) - 5 = 6x - 5$$

b.) $f(x) = 4x^{1/2} - \frac{5}{x}$

$$f'(x) = 4\left(\tfrac{1}{2}x^{-1/2}\right) + 5\left(\tfrac{1}{x^2}\right)$$

$$f'(x) = \frac{2}{\sqrt{x}} + \frac{5}{x^2}$$

c.) $u = \frac{3}{5} x^{5/3} - \frac{5}{3} x^{-3/5}$

$$\frac{du}{dx} = \frac{\cancel{3}}{\cancel{5}} \cdot \frac{\cancel{5}}{3} x^{2/3} - \frac{\cancel{5}}{\cancel{3}} \cdot \frac{-\cancel{3}}{\cancel{5}} x^{-8/5}$$

$$\frac{du}{dx} = x^{2/3} + x^{-8/5}$$

Ex: Find an eqn of the tangent line to $y = \frac{1-4x^2}{x^3}$ at the point on the curve where $x = 1$

$$\left.\frac{dy}{dx}^*\right|_{x=1} = \frac{d}{dx}\left.\left(\frac{1-4x^2}{x^3}\right)\right|_{x=1} = \frac{d}{dx}\left.\left(\frac{1}{x^3} - \frac{4x^2}{x^3}\right)\right|_{x=1}$$

$$= \frac{d}{dx}\left.\left(\frac{1}{x^3} - \frac{4}{x}\right)\right|_{x=1} = \left.\left(\frac{-3}{x^4} + \frac{4}{x^2}\right)\right|_{x=1} = 1$$

$m = 1$ // slope of tangent line

if $x = 1 \Rightarrow y = \frac{1-4}{1} = \frac{-3}{1} = -3$ $\quad (1, -3)$.

Eqn of tangent line to curve at $(1, -3)$ is:

$$y + 3 = 1(x-1) \Rightarrow \boxed{y = x - 4}$$

Thm3 Product Rule

$$\boxed{(fg)'(x) = f'(x)g(x) + g'(x)f(x)}$$