Z-TRANSFORM

Historically, Laplace transforms were used to study signals defined by the solutions to linear ordinary differential equations. Beginning in the 1950's, discrete time signals began to appear. Unfortunately, the Laplace transform is not well suited for the study of discrete time signals and systems. Instead, another transform, called the Z-transform, is used.

1. Direct Z-transform

It is know that discrete-time signal can be obtained by multiplying continuous time signal x(t) by impulse-train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

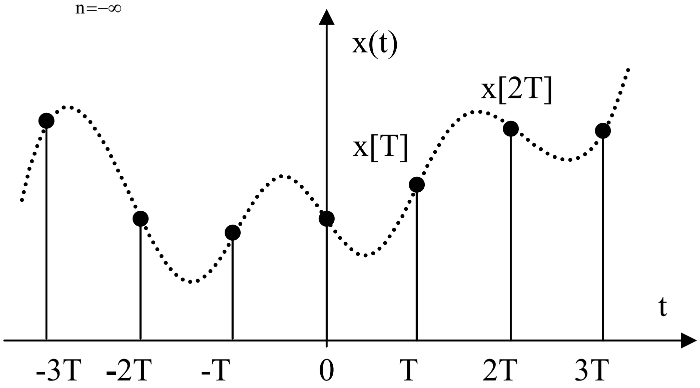


Figure 1

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \tag{1}$$

Using the Laplace transform's shifting property:

$$L\{\delta(t - nT)\} = e^{n}$$

and a short hand notation $e^{sT} = z$, we get

$$Z[x(t)] = X(z) = \sum_{n=-\infty}^{\infty} x(nT)z^{-n}$$

if T=1 we get

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2}$$

Equation (2) is called bilateral or two sided Z-transform. For causal signals the one sided Z-transform uses

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{3}$$

Example 1

Find the Z-transform of the DTS shown in Figure 2

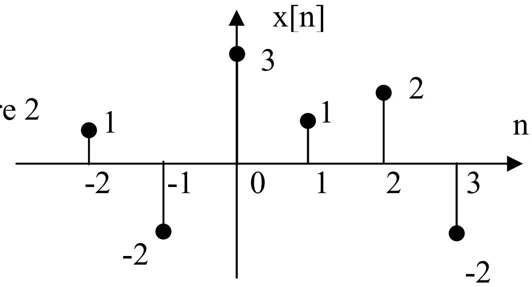


Figure 2

$X(z) = -2z^{-3}+2z^{-2}+z^{-1}+3+-2z^{1}+z^{2}$

Example 2

Find the Z-transform of a causal step function show in Figure 3

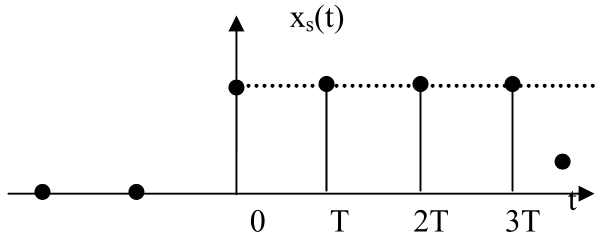


Figure 3

$$X(z)=\sum_{n=0}^{\infty}=1+z^{-1}+z^{-2}+z^{-3}+.......$$

This infinitely long series can also be represented in closed form using :

$$\sum x=1+x+x^2+x^3+.....=\frac{1}{1-x} \qquad \text{if } |x|<1$$

$X(z)=1+z^{-1}+z^{-2}+z^{-3}+.........$ has a closed form

$$X(z)=\frac{1}{1-z^{-1}}=\frac{z}{z-1} \qquad \text{if } |z^{-1}|<1 \text{ or } |z|>1$$

Example 3

Find the Z-transform of the causal exponential function

$$x(t)=e^{-\alpha t} \text{ or } x[n]=e^{-\alpha n}$$

$$x[n]=e^{-\alpha n}=a^n \text{ where } \qquad a=e^{-\alpha}$$

$$X(z)=\sum_{n=0}^{\infty}a^n z^{-n}=\sum_{n=0}^{\infty}(az^{-1})^n=\frac{1}{1-az^{-1}}=\frac{z}{z-a} \quad \text{if } |az^{-1}|<1 \text{ or } |z|>a$$

$$X(z)=\frac{z}{z-e^{-\alpha}}; \quad |z|>e^{-\alpha}$$

Example 4

Find the Z-transform of an anticausal sequences shown in Figure 4

$$x[n]=\begin{cases} -1 & n\le -1 \\ 0 & n\ge 0 \end{cases}$$

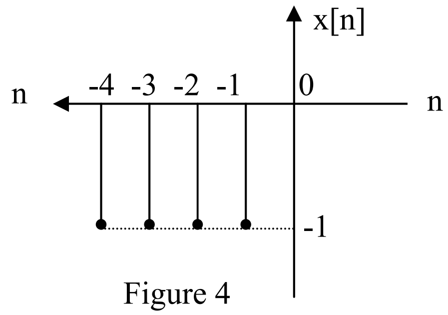


Figure 4

$$X(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n} = \sum_{n=-\infty}^{-1} -z^{-n} = -\left(\sum_{n=0}^{-\infty} z^{-n} - 1\right) = -\left(\sum_{n=0}^{\infty} z^{-n} - 1\right)$$

$$= -\frac{1}{1-z^{-1}} + 1 = \frac{-1+1-z}{1-z} = \frac{z}{1-z}; \qquad \text{if } |z| < 1$$

Computer Study:

M - file ***ztrans.m*** is used to find Z-transform of the time domain function. Consider the following two examples.

Example 5 $x(t)=e^{-2t}$

$$\Rightarrow e^{-2t} = \frac{z}{z-e^{-2}}$$

```
» syms t
» x=exp(-2*t);
» ztrans(x)

ans =

z/(z-exp(-2))
```

Example 6 $X(t)=t \Rightarrow \frac{z}{(z-1)^2}$

```
» syms t
» x=t; ztrans(x)
ans =
z/(z-1)^2
```

2. Region of Convergence of Z-Transform

The domain of values of *z* guaranteeing that a Z-transform of $x[n]$ exists is called the region of convergence, or simply ROC. The ROC constrain of an annular ring in the Z-plane that is centered around the origin. The ROC is an important concept for a variety of reasons. There is no unique relationship between the sequences and their Z-transforms.

The Z-transform for both functions is $X(z) = \frac{z}{z-1}$, but ROC are different. Hence, the Z-transform must always be specified with its ROC. The regions of convergence for the Examples 2 and 4 are given in figures 5 (a) and (b) respectively.

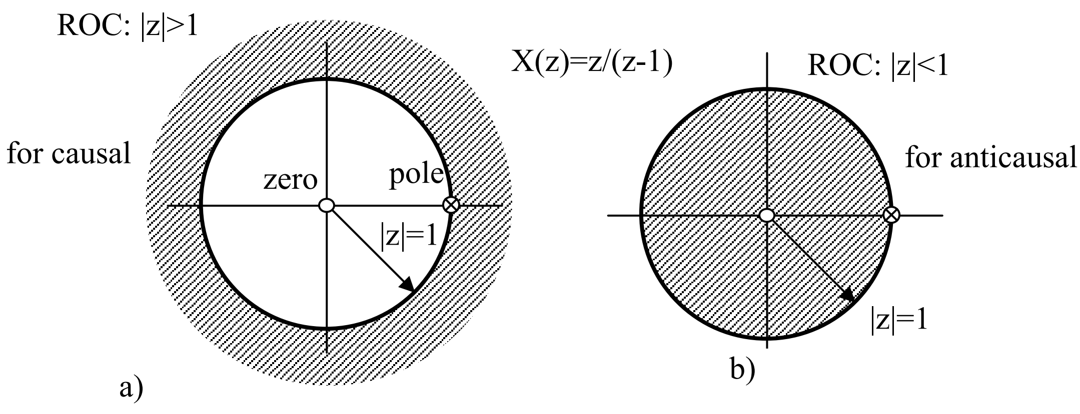


Figure 5

We can readily verify that when $|z|>1$ causal step sequences X(z) convergence where as when $|z|<1$ X(z) diverges. For example, if we let z=2 we find that the series on the RHS of equation adds up to 2:

$$X(z)=1+1/2+(1/2)^2+(1/2)^3+...=2/(2-1)=2$$

as it is clearly a geometric series with a common ration of 1/2 and a first term of 1, giving the sum to infinity of 2/(2-1)=2. On the order hand if z=1/2 (inside the unit circle)the series of equation becomes

$$X(z)=1+1/0{,}5+(1/0{,}5)^2+(1/0{,}5)^3+3...=1+2+4+8+...$$

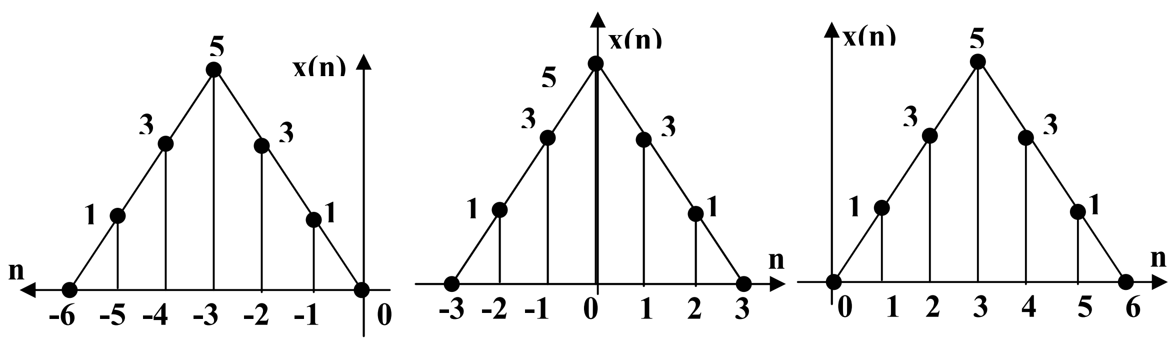
which is seen to be diverging. The region of convergence (hatched) is seen to be bounded by the circle $|z|=1,$ the radius of the pole of X(z). Values of z for which $X(z)=\infty$ are referred to as poles of X(z). Values of z for which X(z)=0 are referred to as the zeros of X(z).

Example 5
Find thez-transform and the region for convergence for each of the discrete-time sequences given in figure 6

(1) The sequence of figure 6 (a) is noncausal, since x(n) is not zero for n<0, but it is of a finite duration. The sequence has values x(-6)=0, x(-5)=1, x(-4)=3, x(-3)=5, x(-2)=3, x(-1)=1 and x(0)=0. from equation 6, the z-transform is given by

$$X_1(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n} = z^5+3z^4+5z^3+3z^2+z$$

It is readily verified that the value of X (z) becomes infinite when $z=\infty$. Trus the ROC is everywhere in the z-plane except at z=∞.



a) b) c)

Figure 6

(2) Again, the sequence in figure 6(b) is not causal. It is of a finite duration, and double sided. The values of the sequence are x(3)=0, x(-2)=1, x(-1)=3, x(0)=5, x(1)=3, x(2)=1 and x(3)=0. from equation 6, the z-transform is given by

$$X_2(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = z^2 + 3z + 5 + 3z^{-1} + z^{-2}$$

It is evident that the value of X(z) is infinite if z=0 or if z=∞. Therefore the region of convergence is everywhere expect at z=0 and z=∞.

(3) figure 6(c) represents a causal, finite duration sequence with values x(0)=0, x(1)=1, x(2)=3, x(3)=5, x(4)=3, x(5)=1 and x(6)=0. the z-transform is given by

$$X_3(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = z^{-1} + 3z^{-2} + 5z^{-3} + 3z^{-4} + z^{-5}$$

In this case , X(z)=∞ for z=0. Trus the region of convergence is everywhere except at z=0.

Example 7

$$x(n) = \left[(1/3)^n + (-1/4)^n\right] u_s[n]$$

Now the procedure continues as:

$$X(z) = \frac{1}{1 - \frac{1}{3} z^{-1}} + \frac{1}{1 + \frac{1}{4} z^{-1}} = \frac{z(2z - \frac{1}{12})}{(z - \frac{1}{3})(z + \frac{1}{4})}$$

For convergence, the individual terms must converge. This means that:

$$\frac{1}{1 - \frac{1}{3} z^{-1}} \longrightarrow \left| \frac{1}{3} z^{-1} \right| < 1$$

$$\frac{1}{1 + \frac{1}{4} z^{-1}} \longrightarrow \left| \frac{1}{4} z^{-1} \right| < 1$$

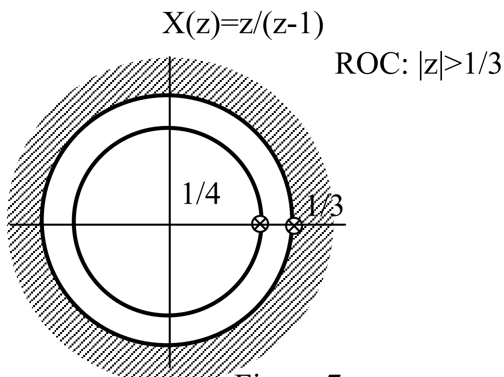


Figure 7

The overlap of two ROC's corresponds to the region of converges of X(z), that is |z|>1/3.

Example 8

Consider the two-sided sequence defined by:

$$u[n] = \alpha^n$$

where α can be a real or complex number, does not have a Z-transform, regardless of the absolute value α. This followed by noting that the Z-transform expression can be rewritten as:

$$U[z] = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n-\infty}^{-1} \alpha^n z^{-n}$$

The first term on the right-hand side of the equation converges for $|z| > |\alpha|$,whereas the second term converges for $|z| < |\alpha|$,and hence, there is no overlap of two ROC's.

Example 9

Determine the ROC of the Z-transform H(z) of the sequence $h[n] = (-0.6)^n Us[n]$.

$$H(z) = \sum_{n=0}^{\infty} (-0.6)^n z^{-n} = \sum_{n=0}^{\infty} (-0.6z^{-1})^n$$

which simplifies to $$H(z) = \frac{1}{1+0.6z^{-1}} = \frac{z}{z+0.6}$$

provided $|z|>0.6$. This implies that the ROC is just outside the circle going through the point z= -0.6 and extending all the way to z=∞, as show in figure. Notice that H(Z) has a zero at z=0 and a pole at z= -0.6.

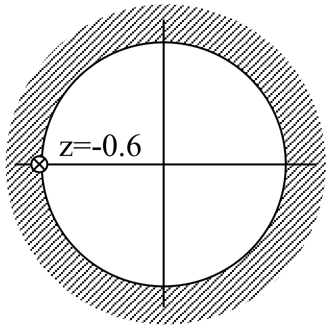


Figure 8

3.Method of Residue

This method of residue is used if original function is represented by s-variable: $F(z) = \sum_{s_i} \left[\text{residues} \frac{F(\nu)}{1-e^{\nu T} z^{-1}} \right]_{\nu=s_i}$

The residue ror simple poles

$$\Gamma_i = \frac{N(s_i)}{D'(s_i)} \cdot \frac{1}{1-e^{-s_i T} z^{-1}}$$

For multiple poles

$$\Gamma_i = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{d\nu^{n-1}} \left\{ (\nu - s_i)^n F(\nu) \frac{1}{1-e^{T\nu} z^{-1}} \right\} \right]_{\nu=s_i}$$

Example 9

$$F(s) = \frac{1}{s^2(s+1)}$$

Using the residues:

$$F(z) = \sum_{\substack{s_1=-1 \\ s_2=0}} \left[\text{residues} \frac{1}{\nu^2(\nu+1)(1-e^{T\nu} z^{-1})} \right]_{\nu=s_1;\, s_2}$$

For simple pole s_1=-1:

$$\Gamma_1 = \left[\frac{1}{D'(\nu)} \frac{1}{1-e^{T\nu} z^{-1}} \right]_{\nu=-1} = \left[\frac{1}{2\nu(\nu+1)+\nu^2} \frac{1}{1-e^{T\nu}} \right]_{\nu-1} = \left[\frac{1}{1-e^{-T} z^{-1}} \right] = \frac{z}{z-e^{-T}}$$

For double pole s_1=0:

$$\Gamma_2 = \frac{d}{d\nu} \left[\nu^2 \frac{1}{\nu^2(\nu+1)} \frac{1}{1-e^{T\nu}} \right]_{\nu=0} = \left[\frac{-(1-e^{T\nu} z^{-1}) + (\nu+1) T e^{T\nu} z^{-1}}{(\nu+1)^2 (1-e^{T\nu} z^{-1})^2} \right]_{\nu=0} =$$

$$\frac{-(1-z^{-1}) + Tz^{-1}}{(1-z^{-1})^2} = -\frac{z}{z-1} + \frac{Tz}{(z-1)^2}$$

$$z\left[\frac{1}{s^2(s+1)}\right] = -\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}$$

Comparison between F(z) and F(s)

$$F(p) = \left[\frac{1}{s^2(s+1)}\right] = -\frac{1}{s-1} + \frac{1}{(s)^2} + \frac{1}{s+1}$$

$$F(z) = z\left[\frac{1}{s^2(s+1)}\right] = -\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}$$

Example 10

Find the Z-transform of $x[t] = \sin 2t$

$$L[\sin 2t] = \frac{2}{s^2+2^2}; \qquad s = \pm 2j$$

$$X(z) = \frac{2}{2\nu}\cdot\frac{1}{1-e^{T\nu}z^{-1}}\bigg|_{\nu=2j} + \frac{2}{2\nu}\cdot\frac{1}{1-e^{T\nu}z^{-1}}\bigg|_{\nu=-2j} = \frac{1}{2j}\left[\frac{1}{1-e^{2jT}z^{-1}} - \frac{1}{1-e^{-2jT}z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{2jT}} - \frac{z}{z-e^{-2jT}}\right] = \frac{1}{2j}\left[\frac{z(z-e^{-2jT}) - z(z-e^{2jT})}{(z-e^{2jT})(z-e^{-2jT})}\right] = \frac{1}{2j}\left[\frac{z^2 - ze^{-2jT} - z^2 + ze^{2jT}}{z^2 - ze^{-2jT} - ze^{2jT} + 1}\right]$$

$$= \frac{1}{2j}\left[\frac{z(e^{2jT} - e^{-2jT})}{z^2 - \frac{2}{2}z(e^{2jT} + e^{-2jT}) + 1}\right] = \frac{z\sin 2T}{z^2 - 2z\cos 2T + 1}$$

$$Z[\cos 2t] = \frac{z\sin 2T}{z^2 - 2z\cos 2T + 1}$$

Computer Study

MATLAB can be used to determine the ROC's of a rational Z-transform. The M-file **[z, p, k] = tf2zp(num, den)** determines the zeros, poles and the gain constant of a rational Z-transform expressed as a ratio of polynomials in descending powers of z. The output files are the column vectors ***z*** and ***p*** containing the zeros and poles of the rational Z-transform, and the gain constant k. The statement **[num, den] = tf2zp (z, p, k)** is used in implement the reverse process. From the zero-pole description, the factored form of the transfer function can be obtained using the function **sos = tf2sos(z, p, k)**. The statement computes the coefficients of each second order factor given as an $L \times 6$ matrix **sos**, where

$$sos = \begin{pmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{21} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{oL} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{pmatrix}$$

where the k-th row contains the coefficients of the numerator and the denominator of the k-th second order factor of the Z-transform G(z):

$$G(z)=\prod_{k=1}^{L}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{a_{0k}+a_{1k}z^{-1}+a_{2k}z^{-2}}$$

The pole-zero plot of the rational Z-transform can also be plotted by using the M-files

zplane (zeros,poles),   zplane (num,den)

It should be noted that the argument zeros and poles must be entered as column vectors, whereas, the argument `num` and `den` needed to be entered as row vectors.

The following example illustrates the application of the above functions.

Example 11 Express the following Z-transform in factored form, plot its poles and zeros, and the n determine its ROC's.

$$G(z)=\frac{2z^2+5z-6}{z^3+8z^2+5z^{-4}}$$

The factored form of the Z-transform is given by:

$$G(z)=\frac{\left(6.232+z^{-3}\right)\left(0.3209+0.8023z^{-1}-0.9627z^{-2}\right)}{\left(1.0000+7.2322z^{-1}\right)\left(1.0000+0.7678z^{-2}-0.5531z^{-3}\right)}$$

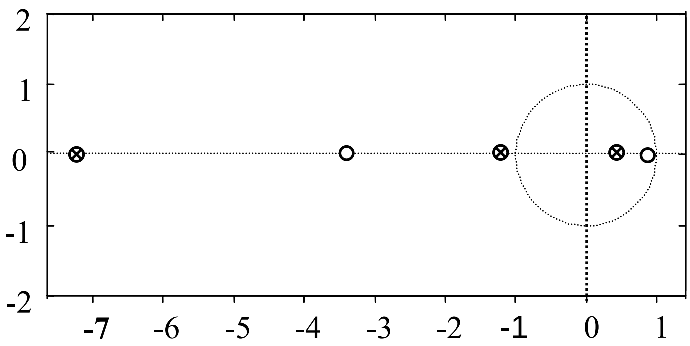


Figure 9

```
» syms z
» G=(2*z^2+5*z-6)/(z^3+8*z^2+5*z-4);
» num=[2 5 -6]; den=[1 8 5 -4];[z,p,k]=tf2zp(num,den)
z =
-3.3860
0.8860
p =
-7.2322
-1.2209
0.4530
k =2
» sos=zp2sos(z,p,k)
sos =
6.2322      0      0   1.0000   7.2322      0
0.3209   0.8023  -0.9627   1.0000   0.7678  -0.5531
» z=[ -3.3860;0.8860]; p=[ -7.2322; -1.2209; 0.4530];
» [num,den]=zp2tf(z,p,k)
num =
0  2.00000000000000  5.00000000000000  -
5.99999200000000
den =
1.00000000000000  8.00010000000000
5.00053868000000  -3.99989621994000
» printsys(num,den)
num/den =
2 z^2 + 5 z - 6
```

The pole-zero configuration developed by the program is shown in Figure 9. From the equation the ROC are:

$R_1: \infty \geq |z| > 7.2323$

$R_2: -7.23 \leq |z| < 1.2209$

$R_3: \quad 1 > |z| \geq 0.453$

4. The properties of the Z-transform

The z-transforms can be represented by following shorthand notations:

$$X(z);\ z[x(t)];\ z[x(s)];\ z[x[n]]$$

1. Linearity:

$$z[ax_1(t)+bx_2(t)] = ax_1(z)+bx_2(z)$$

2. Time shifting:

$$z[x(t+nT)] = z^{-n}X(z)$$
$$Z[x(t+nT)] = z^nX(z)-z^nX(0T)-z^{n-1}X(T)--zX[(n-1)T]$$

3. Frequency shifting:

$$Z[x(s\pm a)] = X(ze^{\pm a})$$

4. Multiplication by t^n:

$$Z[t^nx(t)] = -TZ\frac{d}{dz}[X_1(z)]$$
$$X_1(z) = z[t^{n-1}x(t)]$$
$$Z[tx(t)] = -Tz\frac{d}{dz}X(z)\Big|_{T=1} = -z\frac{d}{dz}X(z)$$

Example 12

Find the Z-transform of x(t) = t

$$x(t)\,|_{t=n} = x[n] = n$$

$$X(z) = \sum_{n=0}^{\infty} nz^{-n}$$

$$X_1(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}$$
$$\frac{dX_1(z)}{dz} = -k\sum_{n=0}^{\infty} z^{-n-1} \Rightarrow -z\frac{dX_1(z)}{dz} = \sum_{n=0}^{\infty} nz^{-n}$$
$$X(z) = -z\frac{dX_1(z)}{dz} = -z\frac{(z-1-z)}{(z-1)^2} = \frac{z}{(z-1)^2}$$
$$X(z) = \sum_{k=0}^{\infty} kz^{-k} = \frac{z}{(z-1)^2}$$

Example 13 Find the Z-transform of $x[n] = n^2u[n]$.

$$X_1(z) = \frac{z}{(z-1)^2}$$

Using the property 4:

$$X(z) = -z\frac{d}{dz}X_1(z) = -z\left(\frac{z2-2z+1-2z^2+2z}{(z-1)^4}\right) = z\left(\frac{z^2-1}{(z-1)^4}\right) = \frac{z(z+1)}{(z-1)^3}$$

An extension to this result is given by $x[k] = ka^k$ for $|a\ |<1$. Letting $x_1[k] = k$ and $x[k] = a^k x_1[k]$, and given knowledge of $X_1(z)$, it fallows that:

$$X(z) = X_1(z/a) = \frac{az}{(z-a)^2}$$

5. The initial value theorem:

$$\mathrm{Lim}_{n\to 0}\ x(nT) = \mathrm{Lim}_{z\to\infty}\ X(z)$$

6. The final value theorem:

$$\mathrm{Lim}_{n\to\infty}\ x(nT) = \mathrm{Lim}_{z\to 1}\ (z\text{-}1)X(z)$$

7. Differentiation:

$$z\left[\frac{d}{d\alpha}x(nT,\alpha)\right] = \frac{d}{d\alpha}F(z,\alpha)$$

8. Integration:

$$z\int_{\alpha_0}^{\alpha_1} x(nT,\alpha)d\alpha = \int_{\alpha_0}^{\alpha_1} x(z,\alpha)d\alpha$$

Example 14

Given $z\left[\frac{1}{s+a}\right] = \frac{z}{z-e^{-at}}$; Find $z\left[\frac{1}{(s+a)^2}\right]$

$$z\left[\frac{1}{(s+a)^2}\right] = z\left[-\frac{d}{d\alpha}\frac{1}{(s+a)}\right] = -\frac{d}{d\alpha}z\left[\frac{-1}{s+a}\right] = -\frac{d}{d\alpha}z\left[\frac{-1}{s+a}\right] = -\frac{d}{d\alpha}\frac{z}{z-e^{-\alpha T}} = \frac{Te^{-\alpha T}}{(z-e^{-\alpha t})^2}$$

9. Convolution:

$$x_1[n] * x_2[n] = x_1(z)\cdot x_2(z); \qquad \text{ROC: } R_1 \cap R_2$$

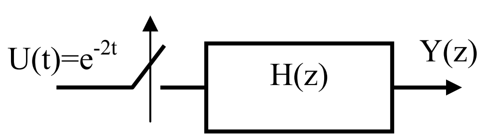
$$\text{where: } x_1[n] \leftrightarrow x_1(z) \qquad \text{ROC: } R_1$$

$$x_2[n] \leftrightarrow x_2(z) \qquad \text{ROC: } R_2$$

Example 15

Find the output signal of the system shown in the Figure 10

$U(z)=z\left[e^{-2t}\right]=\frac{z}{z-e^{-2}}$; $H(z)=\frac{z}{z-1}$



$$Y(z)=\frac{z}{z-1}\cdot\frac{z}{z-e^2} = \frac{z^2}{(z-1)(z-e^{-2})}$$

Figure 10

5. Mapping the s-plane into z-plane

The connection between s-plane and z-plane is interpreted in the figure 11. A mapping rule is based on z=cosωt + jsinωt. Some important mapping are listed below:

1. The points s= ±j2πk (k=0, 1, 2. . .) is mapped to z=exp(j2πk) → cos2πk =1.

2. The point s=±jπ(2k+1) is mapped to z=expj(π(2k+1) → cosπ(2k+1) =-1.
3. The ±jω axis (locus p6) maps onto z6 - the unit circle with the radius z =1. For the values s=±jω, (0≤ω≤2π) , if ω=0, z=exp(0)=1; if ω=π/2→sin π/2=1.
4. The left half of s-plane, for values s=-σ±jω, such that |ω|≤ π) maps to the interior of the unit circle.
5. The right half of s-plane, for values s=σ±jω, such that |ω|≤ π) maps to the exterior of the unit circle.
6. The locus p1, for values s=-σ+jπ/2, (0≤σ≤−∞) maps onto z1. If σ=0, z1=exp(jπ/2)→sin(π/2)=1, if σ=-∞, z1=exp(-∞)=0.
7. The locus p2, for values s=-σ, (0≤σ≤−∞) maps onto z2. If σ=0, z2=exp(jπ/4) →sin(π/4)=0.707, if σ=-∞, z2=exp(-∞)=0.
8. The locus p3, for values s=-σ-jπ/4, (0≤σ≤∞) maps onto z3. If σ=0, z3=exp(jπ/4) →jsin(π/4)=0.707, if σ=-∞, z3=exp(-∞)=0.
9. The locus of p4, for values s=-σ-jσ, (0≤σ≤−∞) maps onto z4. If σ=0, z4=1, if σ=-∞, z4=exp(-∞)=0.
10. The locus p5, for values as s=-σ+jπ, such that 0≤σ≤−∞; maps onto z5. If σ=0, z=-1; If σ=∞, z=0.
10. The locus p7=σ_1- jω), maps onto circle with radius |z6|=$e^{\sigma 1}$. In Figure 11 σ_1=-0.5.

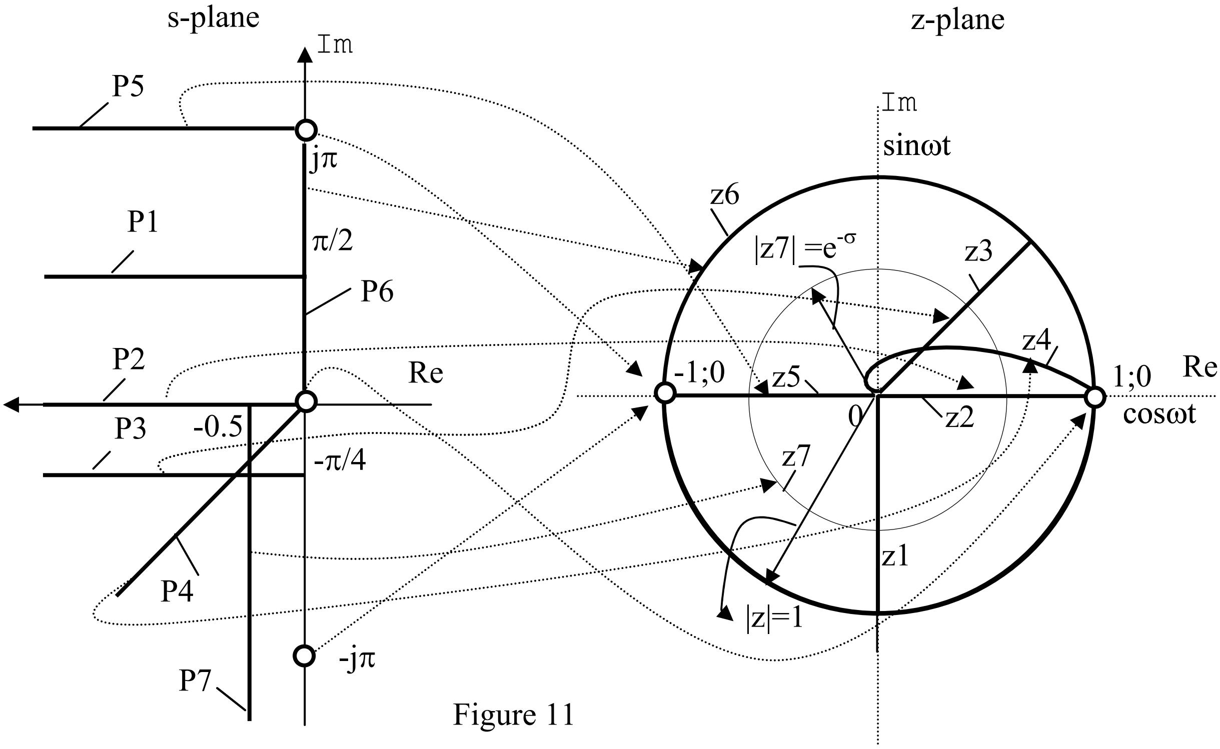
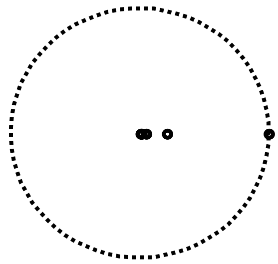


Figure 11

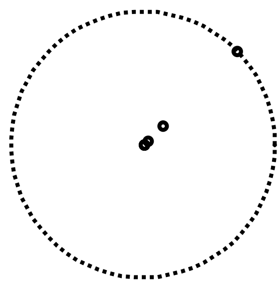
Computer study

M-file **zplane(p_c,T)** convert the points (x_c- in column vectors) in the s-plane into their locations in the z-plain with sampling period (**T**) the unit circle for reference. Each point is represented with a 'o' in the z-plane. The results of some computer mapping of the values of s= σ ± jω, for the interval [0:2*pi] are represented below.

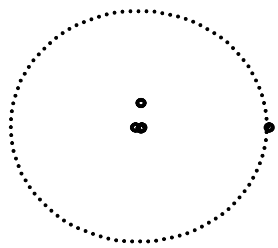
```
» sigma=0:pi/2:2*pi;
» p1=- sigma+sqrt(-1)*pi/2;
» z1=(exp(p1))'

z1 =

   0.0000 - 1.0000i
   0.0000 - 0.2079i
   0.0000 - 0.0432i
   0.0000 - 0.0090i
   0.0       - 0.0019i
```



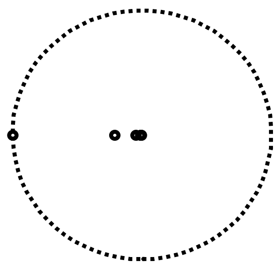
```
» zplane(z1,1)
»   p2=-sigma;
»   z2=(exp(p1))'
z2 =
    1.0000
    0.2079
    0.0432
    0.0090
    0.0019
»    zplane(z2,1)
```



```
» p3=- sigma-sqrt(-1)*pi/4;
» z3=(exp(p3))'
z3 =
   0.7071 + 0.7071i
   0.1470 + 0.1470i
   0.0306 + 0.0306i
   0.0064 + 0.0064i
   0.0013 + 0.0013i
»  zplane(z3,1)
```



```
»  p4=- sigma-sqrt(-1)*sigma;
»  z4=(exp(p4))'
z4 =
   1.0000
   0.0000 + 0.2079i
  -0.0432 + 0.0000i
   0.0000 - 0.0090i
   0.0019 - 0.0000i
» zplane(z4,1)
```



6. INVERSE Z-TRANSFORM

The inverse z-transform(IZT) allows us to recover the discrete-time sequence x(n), given its z-transform. The IZT is particularly useful in DSP work, for example in finding the impluse response of digital filters. Symbolically, the inverse z-transform may be defined as

$$x(n) = Z^{-1}[X(z)] \quad (3)$$

whre X(z) is the z-transform of x(n) and Z^{-1} is the symbol for the inverse z-transform.

Assuming a causal sequence, the z-transform, X(z), in equation 3 can be expanded into a power series as

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \ldots \quad (4)$$

It is seen that the values of x(n) are the cofficients of $z^{-n}\,(n = 0,1,\ldots)$ and so can be obtained directly by inspection. In practice, X(z) is often expressed as a ratio of two polynomials in z^{-1} or equivalently in z:

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_M z^{-M}} = \frac{\sum_{i=0}^{N} b_i z^{-i}}{\sum_{i=0}^{M} a_i z^{-i}} \quad (5)$$

In this form, the inverse z-transform, x(n), may be obtained using one of several methods including the following three:

(1) power series expansion method;
(2) partial fraction expansion method;
(3) residue methode.

Each method has its own merits and demerits. In terms of mathematical rigor, the residue method is perhaps the most elegant. The power series method, however, lends it self most easily to computer implementation.

Computer study

M-file **iztrans.m** is used to find inverse Z-transform.

Example 16

$$X(z) = \frac{z}{z^2 + 5z + 6}$$

$$x[k] = (-2)^n U_s[n] + (-3)^n U_s[n]$$

```
» syms z k
» x=z/(z^2+5*z+6);
» iztrans(x)

ans =

(-2)^k-(-3)^k
```

6.1 Inverse Z-transforms via long division

For causal sequences, the z-transform X(z) can be expended into a power serises in z^{-1}. For a rational X(z), a convenient way to determine the power serise is an expansion by long division.

$$X(z)=\frac{b_n z^n + b_{n-1} z^{n-1} + ... + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0} = c_0 + c_1 z^{-1} + c_2 z^{-2} + ...$$

where $c_0, c_1, c_2.....$ are power serise coefficients.

Example 17 $X(z)=\frac{z}{z^2 - 1.414z^{-1} + 1}$

z	$z^2-1.414z+1$
$z-1.414+z^{-1}$	$z^{-1}+1.414z^{-2}+z^{-3}-z^{-5}...$
$1.414-z^{-1}$	
$1.414-2z^{-1}+1.414z^{-2}$	
$z^{-1}-1.414z^{-2}$	
$z^{-1}-1.414z^{-2}+z^{-3}$	
$-z^{-3}$	
$-z^{-3}+1.414z^{-4}-z^{-5}$	$X[k]=\delta[k-1]+1.414\delta[k-2]+\delta[k-3]+0-\delta[k-5]...$

The inverse of a rational z-transform can also be readily calculated using MATLAB. The function impz can be utilized for this purpose. Three versions of this function are as follows:

- **[h,t]=impz(num,den)**
- **[h,t]=impz(num,den, L)**
- **[h,t]=impz(num,den, L, FT)**

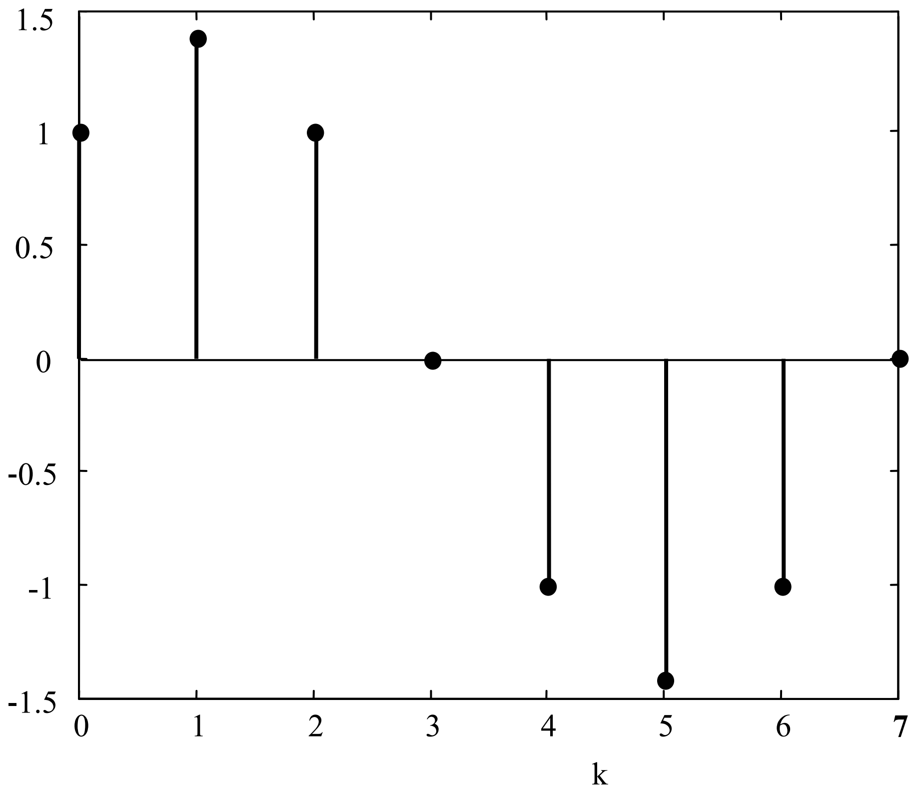
```
» num=[1 0];
» den=[1 -1.414  +1];
» L=8;
 » [x,k]=impz(num,den,L)
x =
1.0000
1.4140
0.9994
-0.0009
-1.0006
-1.4140
-0.9988
0.0017
k =
0
1
2
3
4
5
6
7
» stem(k,x,'fill','k')
```

Where the input data consists of the vector `num` and `den` containing the coefficients of the numerator and the denominator polynomials of the z-transform given in the descending powers of z, the output impulse response vector `h`, and the time index vector `t`. The first form, the length `L` of h is determined automatically by the computer with `t=0:L-1`, whereas in the remaining two forms it is supplied by the user through the input data L. In the last form, the sampling interval is $\frac{1}{FT}$. The default value of *FT* is 1. The following two examples show application $[\mathbf{h},\mathbf{t}]=\mathbf{impz}[\mathbf{num},\mathbf{den}]$ file to and plot power.

Example 18

$$X(z)=\frac{z}{z^2-1.414z+1}$$

Power series coefficiens for $X(z) = \dfrac{z}{z^2 - 1.414z + 1}$



6.2 The Inverse Z-Transform Using Partial Fractions

In this method the z-transform is first expanded into a sum of simle partial fractions. The inverse z-transform of each partial fraction is then obtained from the table of z-transform, And then summed to give the overall inverse z-transform. In many practical cass, the z-transform is given as a ratio of polynomials in z or z^{-1} and has the now familiar form

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_M z^{-M}} \qquad (5a)$$

case1:

Simple poles:

If the poles of X(z) are of first order and N=M, then X(z) can be expanded as

$$X(z) = B_0 + \frac{C_1}{1-p_1 z^{-1}} + \frac{C_2}{1-p_2 z^{-1}} + \ldots + \frac{C_M}{1-p_M z^{-1}} = B_0 + \frac{C_1 z}{z-p_1} + \frac{C_2 z}{z-p_2} + \ldots + \frac{C_M z}{z-p_M} = B_0 + \sum_{k=1}^{M} \frac{C_k z}{z-p_k} \qquad (6)$$

where p_k are the poles of X(z) (assumed distinct), C_K are the partial fraction coefficients and

$$B_0 = b_N / a_N \tag{6.a}$$

The C_k are also know as the residues of X(z);

If the order of the numerator is less than that of the denominator in equation (5a), that is N<M, then B_0 will be zero. If N>M then X(z) must be reduced first, to make $N \le M$, by long division with the numerator and denominator polynomials written in descending powers of z^{-1}. The remainder can then be expressed as in equation 6.

The coefficient, C_K, associate the pole may be obtained by multiplying both sides of equation 6 by $(z - p_k)/z$ and then letting $z = p_k$:

$$C_k = \frac{X(z)}{z}(z - p_k)\Big|_{z=p_k} \tag{7}$$

Case 2:

Multiple poles:

If X(z) contains one or more multiple-order poles then extra terms are required in equation 6 to take this into account. For example, if X(z) contains an mth-order pole at $z = p_k$ the partial fraction expansion must include terms of the form

$$\sum_{i=1}^{m} \frac{D_i}{(z - p_k)^i} \tag{8}$$

The coefficients, D_i, may be obtained from the relationship

$$D_i = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left[(z - p_k)^m \frac{X(z)}{z}\right]_{z=p_k} \tag{9}$$

Evaluation of inverse z-transforms by the partial fraction expansion method is best illustrated by examples.

Example 19

X(z) contains *simple, first- order poles*.

Find the inverse z-transform of the following:

$$X(z) = \frac{z^{-1}}{1 - 0{,}25z^{-1} - 0{,}375z^{-2}}$$

Solution:

For simplicity, we first express the z-tansform in positive power s of z by multiplying the numerator and denominator by z^2:

$$X(z)=\frac{z}{z^2-0{,}25z-0{,}375}=\frac{z}{(z-0{,}75)(z+0{,}5)}$$

X(z) contains first-order poles at z=0,75 and at z=-0,5.Since the order of the numerator is less than the order of the denominator (N<M), the partial fraction expansion has the form

$$X(z)=\frac{z}{(z-0{,}75)(z+0{,}5)}=\frac{C_1 z}{z-0{,}75}+\frac{C_2 z}{z+0{,}5} \qquad (10)$$

To make it easier to find the values of the a_k we divide both sides by z:

$$\frac{X(z)}{z}=\frac{z}{z(z-0{,}75)(z+0{,}5)}=\frac{C_1}{z-0{,}75}+\frac{C_2}{z+0{,}5}$$

To obtain C_1,we simply multiply both sides of equation 10 by (z-0,75) and let z=0,75:

$$C_1=\frac{(z-0{,}75)X(z)}{z}=\frac{(z-0{,}75)}{(z-0{,}75)(z+0{,}5)}\Big|_{z=0{,}75}$$

$$C_1=\frac{1}{z+0{,}5}\Big|_{z=0{,}75}=\frac{1}{0{,}75+0{,}5}=\frac{4}{5}$$

Similarly, C_2 is obtained as

$$C_2=\frac{(z+0{,}5)X(z)}{z}\Big|_{z=-0{,}5}=\frac{z(z+0{,}5)}{z(z-0{,}75)(z+0{,}5)}\Big|_{z=-0{,}5}=\frac{1}{-0{,}5-0{,}75}=-\frac{4}{5}$$

Using the values of C_1 and C_2 in equation 10 we have

$$X(z)=\frac{(4/5)z}{z-0{,}75}-\frac{(4/5)z}{z+0{,}5}$$

The desired inverse z-transform, X(n), is the sum of the two inverse z-transforms:

$$X(n)=\frac{4}{5}\left[(0{,}75)^n-(-0{,}5)^n\right] \text{, n>0}$$

Complex conjugate poles

Example 20

X(z) contains first-order , complex conjugate poles. Find the discrete-time signal, x(n), represented by the following z-transform using the partial fraction expansion method:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0{,}356z^{-2}}$$

Solution:

First, X(z)is expressed in positive powers of z:

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^{2+} + 2z + 1}{z^2 - z + 0{,}3561}$$

The poles of X(z) are found by solving the quadratic $D(z) = z^2 - z + 0{,}3561 = 0$ using the formulae

$$p_1 = \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

$$p_2 = \frac{-b - (b^2 - 4ac)^{\frac{1}{2}}}{2a} \qquad (11)$$

where a and b are the coefficients of z^2 and z, respectively, and c is the constant term. With a=1, b=-1, and c=0,3561 the poles are

$$p_1 = \frac{-1 + (1 - 4x0{,}3561)^{\frac{1}{2}}}{2} = 0{,}5 + 0{,}3257j = re^{j\theta}$$

$$p_2 = p_1^* = 0{,}5 - 0{,}3257j = re^{-j\theta}$$

where r=0,5967 and $\theta = 33{,}08°$.thus we can express X(z) in terms of its poles:

$$X(z) = \frac{z^2 + 2z + 1}{(z - p_1)(z - p_1^*)}$$

since the numerator and denominator of X(z) are of the same order, the partial fraction expension has the form

$$\frac{X(z)}{z} = \frac{B_0}{z} + \frac{C_1}{z - p_1} + \frac{C_2}{z - p_1^*} \qquad (12)$$

From equation 6(a) B_0=1/0,3561=2,8082. To find C_1, we multiply both sides of equation 12 by $z-p_1$ and then let $z=p_1$:

$$\frac{(z-p_1)X(z)}{z}=\frac{B_0(z-p_1)}{z}+C_1+\frac{C_2(z-p_1)}{z-p_2}\Big|_{z=p_1}$$

thus

$$C_1=\frac{(z-p_1)X(z)}{z}=\frac{\cancel{(z-p_1)}(z^2+2z+1)}{z\cancel{(z-p_1)}(z-p_2)}\Big|_{z=p_{1=re^{j\theta}}}=\frac{(re^{j\theta})^2+2re^{j\theta}+1}{re^{j\theta}(re^{j\theta}-re^{-j\theta})}\tag{13}$$

Where r=0,5967, $\theta=33{,}08°$. After some manipulation and simplification we have

$$C_1=\frac{2{,}1439+0{,}97719j}{-0{,}2122+0{,}3257j}=-0{,}9040999-5{,}992847j=6{,}06066<-98{,}58°$$

since p_1 and p_2 are complex conjugate pairs then

$$C_2=C_1^*=-0{,}9040999+5{,}992847j=6{,}06066<98{,}58°$$

thus the z-transform can be expressed as (from equation 12)

$$X(z)=2{,}8082+\frac{C_1z}{z-p_1}+\frac{C_2z}{z-p_1^*}\tag{14}$$

where

$p_1=0{,}5+0{,}3257j$ $\qquad$ $p_2=0{,}5-0{,}3257j$

$C_1=-0{,}9041-5{,}59928j$ $\qquad$ $C_2=-0{,}9041+5{,}59928j$

From the z-transform table, the inverse z-transform of the terms on the right-hand sides of equation 14 is

$$Z^{-1}(2{,}8082)=2{,}8082u(n)$$

$$Z^{-1}\left[\frac{C_1z}{z-p_1}+\frac{C_2z}{z-p_1^*}\right]=2*6{,}06066(0{,}5967)^n\cos(33{,}08n-98{,}58°)$$

$$=12{,}1213(0{,}5967)^n\cos(33{,}08n-98{,}58°)$$

thus the dicrete-time signal becomes

$x(n)=2,8082u(n)+12,1213(0,5967)^n \cos(33,08n-98,58°)$, $\quad n\geq 0$

A useful check for partial fraction results is to compute the values of x(n) for n=0,1,2 (say) and then to compare these with values obtained by the power series method.
For example, from the expression for x(n) we find that

x(0)=2,8082-1,80838=1; x(1)=2,99959=3; x(2)=3,6436

Example 21

X(z) contains a second-order pole. Find the discrete-time sequence, x(n), with the following z-transform:

$$X(z)=\frac{z^2}{(z-0,5)(z-1)^2}$$

solution:

X(z) has a first-order pole at z=0,5 and a second-order pole at z=1. In this case, the partial fraction expansion has the form

$$X(z)=\frac{Cz}{z-0,5}+\frac{D_1 z}{z-1}+\frac{D_2 z}{(z-1)^2} \qquad (15)$$

To obtain C, we processed as before and multiply both sides of X(z) by (z,-0,5), set (z=0,5) and evaluate the expression

$$C=\frac{\cancel{(z-0,5)}z^2}{z\cancel{(z-0,5)}(z-1)^2}\Big|_{z=0,5}=0,5/(0,5-1)^2=2$$

To obtained D_1 we used equation 9, with i=1 and m=2. thus

$$D_1=\frac{d}{dz}\left[\frac{(z-1)^2X(z)}{z}\right]_{z=1}=\frac{d}{dz}\left[\frac{\cancel{(z-1)^2}z^2}{z(z-0,5)\cancel{(z-1)^2}}\right]_{z=1}=\frac{d}{dz}\left(\frac{z}{z-0,5}\right)_{z=1}=\frac{z-0,5-z}{(z-0,5)^2}\Big|_{z=1}=2$$

Similarly, D_2 is obtained from equation 9 by letting i=2 and m=2;

$$D_2=\frac{(z-1)^2X(z)}{z}\Big|_{z=1}=\frac{\cancel{(z-1)^2}z^{\cancel{2}}}{\cancel{z}(z-0,5)\cancel{(z-1)^2}}\Big|_{z=1}=1/(1-0,5)=2$$

Combining the results, X(z) become

$$X(z)=\frac{2z}{z-0,5}-\frac{2z}{z-1}+\frac{2z}{(z-1)^2}$$

The inverse z-transform of each term on the right-hand side is obtained and then summed to give X(n):

$$X(n) = 2(0,5)^n - 2 + 2n = 2 + \left[(n-1) + (0,5)^n\right], \qquad n \geq 0$$

Consider the following three cases:

1) $|z| > 1$

$$g[n] = 2(0.5)^n \mu[n] - 2\mu[n] + 2n\mu[n]$$

2) $|z| < \frac{1}{2}$

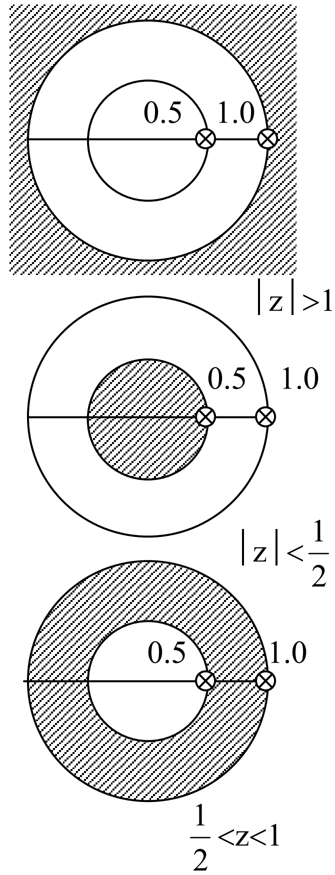
$$\mu(n) \rightarrow \mu[-n-1]$$

$$g[n] = -2(0.5)^n \mu[-n-1] + 2\mu[-n-1] - 2n\mu[-n-1]$$

3) $\frac{1}{2} < z < 1$

$$g[n] = 2(0.5)^n \mu[n] + 2\mu[n-1] - 2n\mu[-n-1]$$

μ(n) – unit step function



6.3. Residue Method

In this method the Inverse z-transform is obtained by evaluating contour integral

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} z^{n-1} X(z) dz$$

For rational polynomials the contour integral is evaluated using a fundamental results in complex variable theory known as Gaushy' residue theorem

$$x(n) = \sum \left[\text{residues of } X(z) z^{n-1} \text{at the poles inside } C\right]$$

The residue of $z^{n-1} X(n)$ at the multiple pole p_k is given by

$$\text{Res}\left[F(z), p_k\right] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - p_k) F(z)\right]_{z=p_k} \qquad (17)$$

Where F(z)= $z^{n-1} X(z)$,m is the order of the pole at p^k and Res $\left[F(z), p_k\right]$ is the residue of F(z) at $z = p_k$.

The residue of simple pole is defined as

$$\text{Res }\left[F(z), p_k\right] = (z - p_k)F(z) = (z - p_k)z^{n-1}X(z)\Big|_{z=p_k} \qquad (18)$$

Simple poles.

Example 22

Find, using the residue method, the discrete- time signal corresponding to the following z-transform:

$$X(z) = \frac{z}{(z-0{,}75)(z+0{,}5)}$$

Solution:

This problem is the same as example 19. In factored form, X(z) is given by

$$X(z) = \frac{z}{(z-0{,}75)(z+0{,}5)}$$

If we let $F(z) = z^{n-1}X(z)$ then

$$F(z) = \frac{z^{n-1}z}{(z-0{,}75)(z+0{,}5)} = \frac{z^n}{(z-0{,}75)(z+0{,}5)}$$

F(z) has poles at $z_1 = 0{,}75$ and $z_2 = -0{,}5$. From equation 18, the inverse z-transform is by given

$$X(n) = \text{Re}\,s\left[F(z), 0.75\right] + \text{Re}\,s\left[F(z), -0.5\right]$$

since the poles are first order, equation 18 will be used. Thus

$$\text{Re}\,s\left[F(z), 0.75\right] = (z-0{,}75)F(z)\Big|_{z=0{,}75} = \frac{\cancel{(z-0{,}75)}z^n}{\cancel{(z-0{,}75)}(z+0{,}5)}\Big|_{z=0{,}75} = \frac{(0{,}75)^n}{0{,}75+0{,}5} = \frac{4}{5}(0{,}75)^n$$

$$\text{Re}\,s\left[F(z), -0.5\right] = (z+0{,}5)F(z)\Big|_{z=-0{,}5} = \frac{\cancel{(z+0{,}5)}z^n}{(z-0{,}75)\cancel{(z+0{,}5)}}\Big|_{z=-0{,}5} = -\frac{4}{5}(-0{,}5)^n$$

The inverse z-transform is sum of the residues at z=0,75 and at z=-0,5:

$$\text{X(n)} = \frac{4}{5}\left[(0{,}75)^{\text{n}} - (-0{,}5)^{\text{n}}\right]$$

which is identical to the result obtained by the partial fraction expansion.

Complex conjugate poles:

Examle 23

The poles of X(z) are complex conjugate poles. Find the inverse z-transform, sing the residue method, given the following z-transform:

$$X(z) = \frac{z^2 + 2z + 1}{z^2 - z + 0{,}3561}$$

solution:

In factored form X(z) is given as:

$$X(z) = \frac{z^2 + 2z + 1}{(z - p_1)(z - p_2)}$$

when $p_1 = 0{,}5 + 0{,}3557j$ and $p_2 = 0{,}5 - 0{,}3557j$, that is $p_2 = p_1^*$. To find the inverse z-transform we evaluate the residues of F(z), where in this case

$$F(z) = z^{n-1}X(z) = \frac{z^{n-1}(z^2 + 2z + 1)}{z^2 - z + 0{,}3561} = \frac{z^n(z^2 + 2z + 1)}{z^2 - z + 0{,}3561} = \frac{z^n(z^2 + 2z + 1)}{z(z^2 - z + 0{,}3561)}$$

F(z) has the same poles as X(z), that is at $z = p_1$ and $z = p_2$, plus apole at z=0 when n=0. All the poles lie inside the contour. The pole at z=0 does not exist for n>0 and so we need to consider the two cases separately.

When n=0, F(z) reduces to

$$F(z) = \frac{z^2 + 2z + 1}{z(z^2 - z + 0{,}3561)}$$

and

$$x(0) = \mathrm{Re}\,s[F(z),0] + \mathrm{Re}\,s[F(z), p_1] + \mathrm{Re}\,s[F(z), p_2]$$

Therefore

$$\mathrm{Re}\,s[F(z),0] = zF(z)\big|_{z=0} = \frac{z(z^2 + 2z + 1)}{z(z^2 - z + 0{,}3561)}\Big|_{z=0} = 1/0{,}3561 = 2{,}8082$$

$$\mathrm{Re}\,s[F(z), p_1] = (z - p_1)F(z)\big|_{z=p_1} = \frac{\cancel{(z - p_1)}(z^2 + 2z + 1)}{z\cancel{(z - p_1)}(z - p_2)} = \frac{(re^{j\theta})^2 + 2re^{j\theta} + 1}{re^{j\theta}(re^{j\theta} - re^{-j\theta})}$$

where r=0,5967 and $\theta = 33{,}08°$. Noting that this expression is identical to that of equation 13, we can write

$$\operatorname{Res}[F(z), p_1] = -0{,}9041 - 5{,}9928j$$

since p_1 and p_2 are complex conjugate pairs then

$$\operatorname{Res}[F(z), p_2] = -0{,}9041 + 5{,}9928j$$

thus

$$x(0) = \operatorname{Res}[F(z),0] + \operatorname{Res}[F(z), p_1] + \operatorname{Res}[F(z), p_2]$$
$$= 2{,}8082 - 0{,}9041 - 5{,}9928j - 0{,}9041 + 5{,}9928j = 1$$

when n>0, the pole at z=0 vanishes and we have

$$F(z) = \frac{z^n(z^2 + 2z + 1)}{z(z^2 - z + 0{,}3561)}$$

$$X(n) = \operatorname{Res}[F(z), p_1] + \operatorname{Res}[F(z), p_2]$$

$$\operatorname{Res}[F(z), p_1] = (z - p_1)F(z)\Big|_{z=p_1} = \frac{\cancel{(z-p_1)}z^n(z^2 + 2z + 1)}{z\cancel{(z-p_1)}(z - p_2)}\Bigg|_{z=p_1} = \frac{(re^{j\theta})^n\left[(re^{j\theta})^2 + 2re^{j\theta} + 1\right]}{re^{j\theta}(re^{j\theta} - re^{-j\theta})}$$

where r=0,5967 and $\theta = 33{,}08°$. Noting that this expression is similar to equation 4.24 , we can write

$$\operatorname{Res}[F(z), p_1] = (0{,}5967e^{j33{,}08})^n(6{,}06066e^{-j98{,}58})$$
$$= 6{,}06066(0{,}5967)^n\left[\cos(33{,}08n - 98{,}58) + j\sin(33{,}08n - 98{,}58)\right]$$

Since p_2 and p_1 are complex conjugate pole pairs we can write

$$\operatorname{Res}[F(z), p_2] = 6{,}06066(0{,}5967)^n\left[\cos(33{,}088n - 98{,}58) - j\sin(33{,}08n - 98{,}58)\right]$$

Thus

$$X(n) = \operatorname{Res}[F(z), p_1] + \operatorname{Res}[F(z), p_2] = 12{,}1213(0{,}5967)^n\cos(33{,}08n - 98{,}58°), \quad n>0$$

which checks with the results for the partion fraction expansion.

X(z) contains a second-order pole.

Example 24

Find the discrete-time sequence, x(n), with the following z-transform:

$$X(z) = \frac{z^2}{(z - 0{,}5)(z - 1)^2}$$

solution:

This example is teh same as example under partial fraction expansion. According to the residue method the discrete-time sequence is given by

$$X(n) = \sum_{k=1}^{M} \mathrm{Re}\, s[F(z, p_k)]$$

where

$$F(z) = z^{n-1} X(z) = \frac{z^{n+1}}{(z-0{,}5)(z-1)^2}$$

F(z) has a imple pole at z=0,5 and a second order pole at z=1; thus x(n) is given by

$$x(n) = \mathrm{Re}\, s[F(z), p_1] + \mathrm{Re}\, s[F(z), p_2]$$

$$\mathrm{Re}\, s[F(z), 0.5] = \frac{(z-0{,}5)z^{n+1}}{(z-0{,}5)(z-1)^2} = \frac{z^{n+1}}{(z-1)^2}\Big|_{z=0{,}5} = 0{,}5(0{,}5)^n/(0{,}5)^2 = 2(0{,}5)^n$$

$$\mathrm{Re}\, s[F(z), 1] = \frac{d}{dz}\left[\frac{(z-1)^2 z^{n+1}}{(z-0{,}5)(z-1)^2}\right] = \frac{(z-0{,}5)(n+1)z^n - z^{n+1}}{(z-0{,}5)^2}\Big|_{z=1} = [(0{,}5)(n+1)-1]/(0{,}5)^2 = 2(n-1)$$

Combining the results, we have

$$X(n) = 2[(n-1) + (0{,}5)^n]$$

which is the same result as for the partial fraction expansion method.

Example. Solve using *Matlab*:

$$H(z) = \frac{18z^3}{18z^3 + 3z^2 - 4z - 1}$$

$$H(z) = \frac{0.24}{1+0.33z^{-1}} + \frac{0.4}{(1+0.33z^{-1})^2} + \frac{0.36}{z-0.5}$$

```
» num=[18]; den=[18 3 -4 -1];
» [r,p,k]=residuez(num,den)
r =
   0.2400
   0.4000
   0.3600
p =

  -0.3333
  -0.3333
   0.5000
k =[]
» [num,den]=residuez(r,p,k)
num =
   1.0000    0.0000    0.0000
den =
   1.0000    0.1667   -0.2222   -0.0556
```

Using the numerator and the denominator coefficients we have:

$$X(z) = \frac{z^3}{z^3 + 0.1667z^2 - 0.2222z - 0.0556}$$

It can be seen that the coefficients will be same as in the equation of the question if we multiply each coefficient by 18.

7. Comparison of the inverse Z-Tra

We have discussed in some detail three methods of obtaining the inverse z-transform:

The power series, partial fraction expansion and the residue methods. A limitation of the power series method is that is does not lead to a closed form solution, but it is simple and lends itself to computer implementation. However, because of its recursive nature care should be taken to minimize possible build-up of numerical errors when the number of data points in the inverse z-transform is large, for example by using double precision.

Both the partial fraction expansion and the residue methods lead to closed form solution. The main disadvantage with both methods is the need to factorize the denominator polynomial, that is finding the poles of X(z) . If the order of X(z) is high finding the poles of X(z), if X(z) is not in factored form, is quite a difficult task .Both methods may also involve high-order differentiation if X(z) contains multiple-order poles.

Clearly, if closed form solution is required then the partial fraction or residue methods is the most appropriate. The partial fraction method is particularly useful in generating the coefficients of parallel structures for digital filters. The residue methods widely used in the analysis of quantization errors in discrete-time system.

Z-Transform Table

Functions	Z-Transform	Functions	Z-Transform
$\delta(t)$	1	$1 - e^{-at}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
$\delta(t - nT)$	z^{-n}	$t\,e^{-at}$	$\frac{Tze^{-at}}{(z-e^{-aT})^2}$
$U_S(t)$	$\frac{z}{z-1}$	$\sin \omega t$	$\frac{z\sin\omega T}{z^2-2z\cos\omega t+1}$
t	$\frac{Tz}{(z-1)^2}$	$\cos \omega t$	$\frac{z^2-z\cos\omega T}{z^2-2z\cos\omega T+1}$
$\frac{1}{2}t^2$	$\frac{1}{2}T^2\frac{z(z+1)}{(z-1)^3}$	$e^{-at}\sin\omega t$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
e^{-at}	$\frac{z}{z-e^{-at}}$	$e^{-at}\cos\omega t$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
a^k	$\frac{z}{z-a}$		