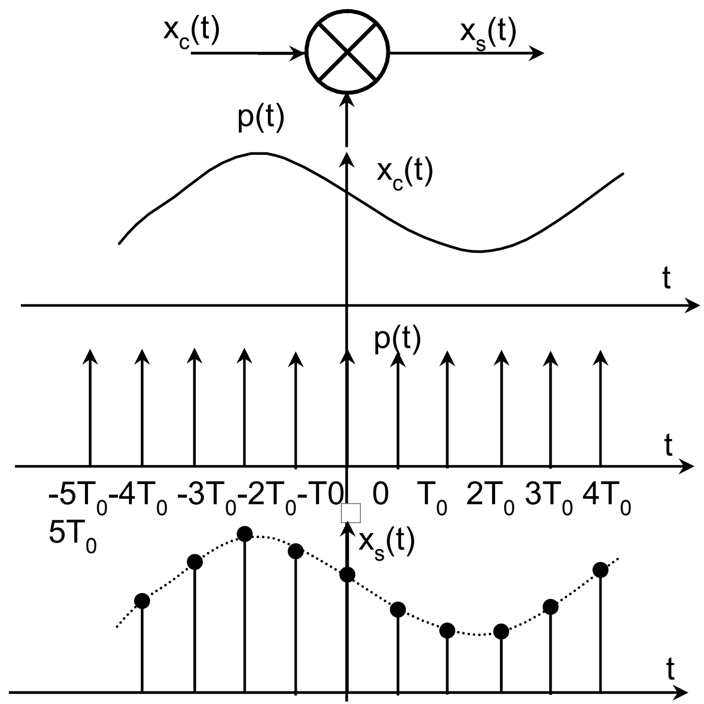
SAMPLING AND RECONSTRUCTION OF SIGNALS

The sampling theorem states that to reconstruct any analog signal from its samples, the sampling frequency ω_0 must be at least twice the signal's maximum frequency $\boldsymbol{\omega_m}$:

$$\boldsymbol{\omega_0 \geq 2\omega_m}$$

Sampling is a presentation of the continuous-time signal $x_c(t)$ by a series of samples $x[nT_0]$.
Consider a signal $x_s(t)$ defined as the product of two signals: $x_c(t)$-is an original signal and p(t) -is a periodic impulse train or Dirac distribution.



$x_s(t) = x_c(t)p(t)$

$$p(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_0)$$

$$x_s(t) = x_c(t)p(t) = \sum_{k=-\infty}^{+\infty} x_c(t)\delta(t - kT_0) = \sum_{k=-\infty}^{+\infty} x_c(kT_0)\delta(t - kT_0)$$

The spectrum of the Dirac distribution p(t) is itself a periodic train.

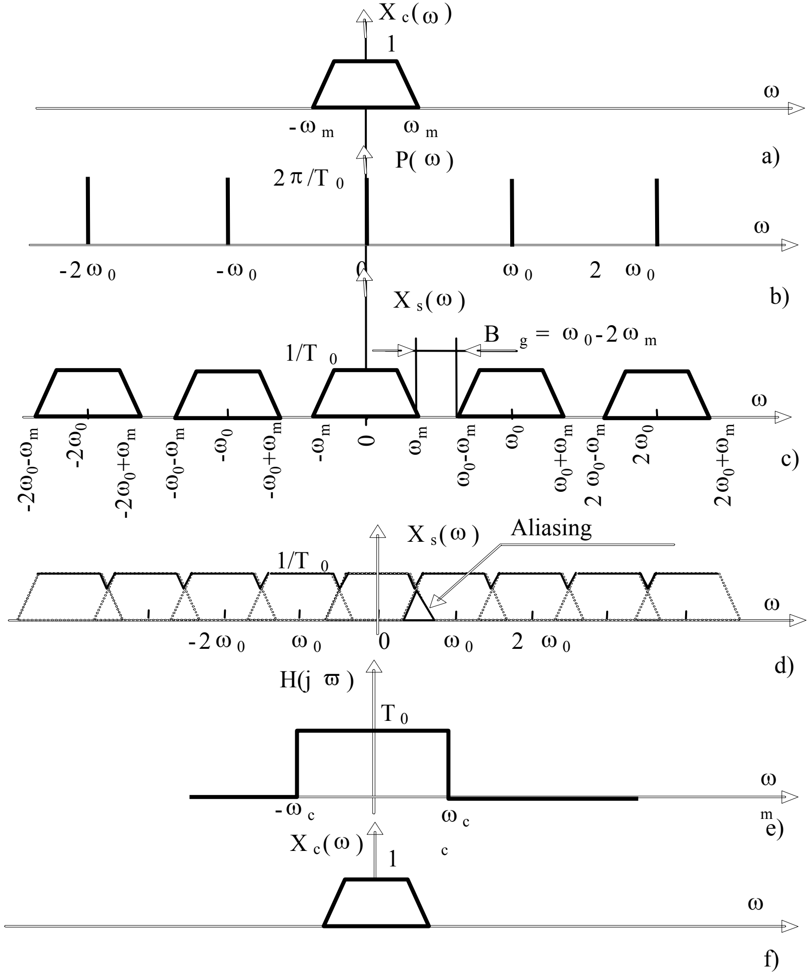
$$P(\omega) = \frac{2\pi}{T_0} \sum_{-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

The spectrum $X_s(\omega)$ of the output signal $x_s(t)$

$$X_s(\omega) = \frac{1}{2\pi}[X_c(\omega) * P(\omega)] = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_0)$$

The spectrum of the signals are shown in Figure 5.2; where $X_c(\omega)$ is arbitrary.
The distance between two adjacent replicated spectra is called a guard band

$$B_g = \omega_0 - 2\omega_m.$$



X c(ω)
1
ω
-ω m
ω m
a)
P(ω)
2π/T 0
ω
-2ω 0
-ω 0
0
ω 0
2 ω 0
b)
X s(ω)
B g = ω 0-2ω m
1/T 0
ω
-2ω0-ωm
-2ω0
-2ω0+ωm
-ω0-ωm
-ω0
-ω0+ωm
-ωm
0
ωm
ω0-ωm
ω0
ω0+ωm
2 ω0-ωm
2ω0
2ω0+ωm
c)
X s(ω)
Aliasing
1/T 0
ω
-2ω 0
ω 0
0
ω 0
2 ω 0
d)
H(j ϖ)
T 0
ω
-ω c
ω c
e)
X c(ω)
1
ω
f)

Figure 5.2

Example 5.1 A band limited signal has a bandwidth equal to 200 Hz. What sampling rate should be used to guarantee a guard band of 100 Hz.
Solution: Fm=200 Hz; Bg=100 Hz. Bg= F0-2Fm; 100=F0-2x100; F0=300 Hz.

The following three cases present practical interest:

Under sampling: $\omega 0 < 2\omega m$

Nyquist rate: $\omega 0 = 2\omega m$

Over sampling: $\omega 0 > 2\omega m$

From Figure 5.2 it is evident that when $\omega 0-\omega m>\omega m$ or $\omega 0>2\omega m$ the spectrum of $Xs(\omega)$ don't overlap (see Figure 5.2 (c)). and consequently it can be recovered from its samples with ideal low-pass filter having a frequency response $H(j\omega)$ (see Figure 5.2 (e)). If $\omega 0>2\omega m$, output of the filter corresponds to $Xc(\omega)$ (see Figure 5.2 (f)). If $\omega 0>2\omega m$ does not hold , i.e $\omega 0<2\omega m$ the spectrum $Xs(\omega)$ overlap(see Figure 5.2 (d)) and xc(t) is not recoverable by low pass filtering because of side-band distortion. This high frequency distortion is called an aliasing.

Reconstruction of a Bandlimited Signal From Its Samples

According to the sampling theorem, samples of a continuous-time band limited signal taken frequently enough are sufficient to represent the signal exactly in the sense that the signal can be recovered from the samples. Impulse train modulation provides a convenient means for understanding the process of reconstructing the continuous-time bandlimited signal from its samples.

If the conditions of the sampling theorem are met and if the modulated impulse train is filtered by an appropriate low-pass filter, then the Fourier transform of the filter output will be identical to the Fourier transform of the original continuous-time signal $x_c(t)$, and thus the output of the filter will be $x^*_c(t)$. If $x_c(nT_0)$ is the input to an ideal low-pass continuous time filter with frequency response $H_r(j\omega)$ and impulse response $h_r(t)$, them the output of the filter will be

$$x^*_c(t) = \sum_{n=-\infty}^{\infty} x_c[nT_0]\, h_r[t - nT_0] \qquad (5.1)$$

A block diagram representation of this signal reconstruction process is shown in Figure 5.3.

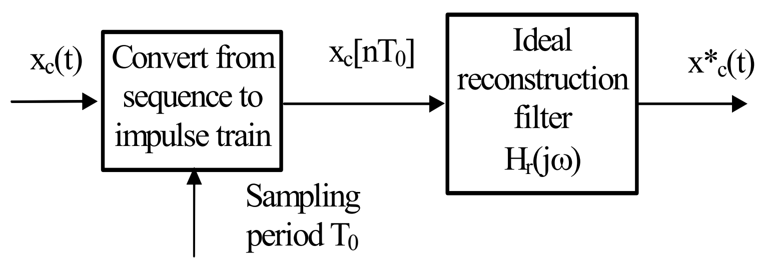
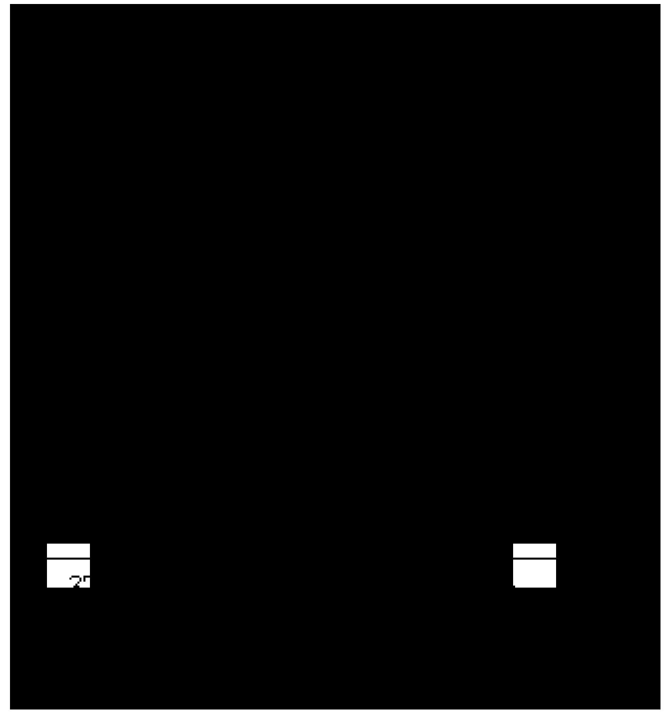


Figure 5.3

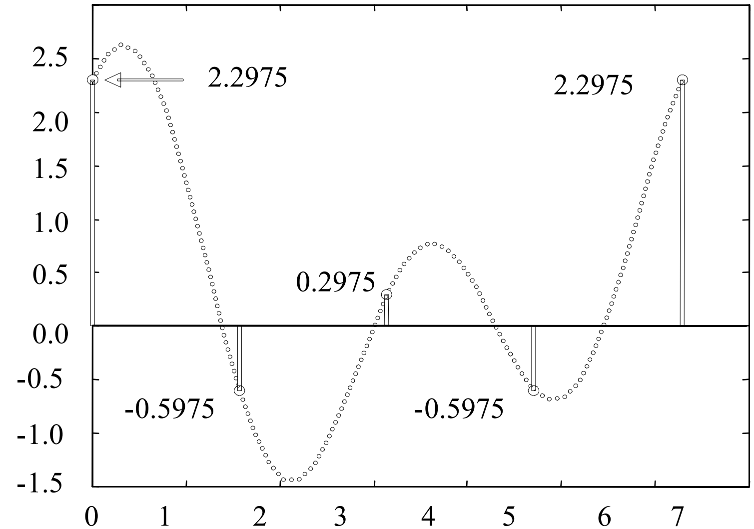
$$H(\omega) = \begin{cases} T0 & \text{if} \quad |\omega| < \omega c \\ 0 & \text{if} \quad |\omega| > \omega c \end{cases}$$



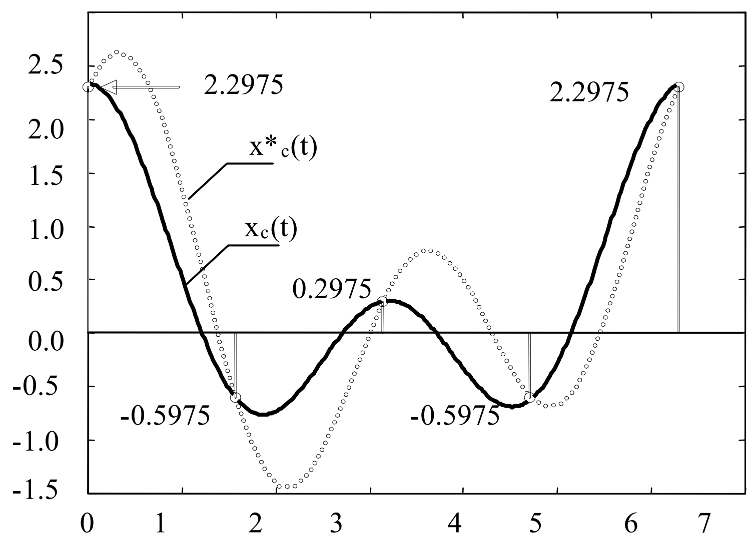
$$h_r(t) = \frac{T_0}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{T_0}{2\pi}\frac{1}{jt}\left[e^{j\omega t}\right]\Big|_{-\omega_c}^{\omega_c} =$$

$$= \frac{T_0}{\pi t}\frac{e^{j\omega_c t} + e^{j\omega_c t}}{2j} = \frac{\sin \pi t / T_0}{\pi t / T_0}$$

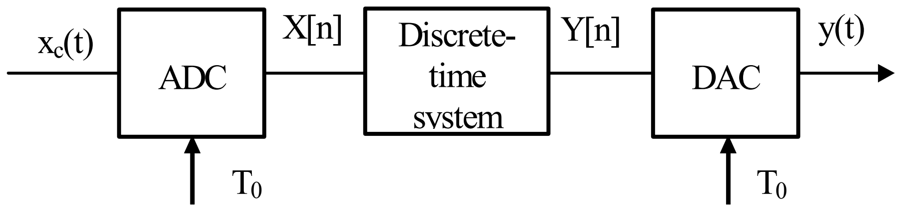
```
» t=0:2*pi/40:2*pi;
» s=0.35+sin(t+pi/2)+
1.34*sin(2*t+pi/4);
»  plot(t,s)
»hold on
» t=0:2*pi/4:2*pi;
» s=0.35+sin(t+pi/2)+
1.34*sin(2*t+pi/4);
» stem(t,s,'fill','k')
```




```
» sxms t k
»x 1=sin(2*(t  -(4*k*pi/2)))/(2*(t   -(4*k*pi/2)));          z1=sxmsum(x1,k,     -2,2);
»  x2=sin(2*(t  -(4*k+1)*pi/2))/(2*(t   -(4*k+1)*pi/2)); z2=sxmsum(x2,k,     -2,2);
»  x3=sin(2*(t  -(4*k+2)*pi/2))/(2*(t   -(4*k+2)*pi/2)); z3=sxmsum(x3,k,     -2,2);
»  x3=sin(2*(t  -(4*k+3)*pi/2))/(2   *(t-(4*k+3)*pi/2)); z4=sxmsum(x3,k,     -2,2);
»  x3=sin(2*(t  -(4*k+4)*pi/2))/(2*(t   -(4*k+4)*pi/2)); z5=sxmsum(x3,k,     -2,2);
» x*c=2.2975*z1   -0.5975*z2+0.2975*z3   -0.5975*z4; ezplot(x*c,[0 2*pi])
```



A major application of discrete-time systems is the processing of continuous-time signals. This is accomplished by a system of the general form depicted in Figure 5.8



ADC- Analog-to-digital converter; DAC- Digital –to-analog converter

Sampling Interval and Lagrange Approximation

Interpolation means to estimate a missing function value by taking weighted average values at neighboring points.
The general form of Lagrange approximation passing true N+1 points $(t_0, x_0),..(t_n, x_n)$ is defined as

$$P_{N,K}(x) = \sum_{K=0}^{N} x_K L_{N,K}(t)$$

Where $L_{N,K}(x)$ are called Lagrange coefficient polynomials.

$$L_{N,K}(t) = \frac{(t - t_0)...(t - t_{K-1})(t - t_{k+1})...(t - t_N)}{(t_K - t_0)...(t_K - t_{K-1})(t_K - t_{K+1})...(t_K - t_N)} \qquad (5.4)$$

The Lagrange polynomial passing true the 2 points (t_1,x_1) and (t_2,x_2) is linear interpolation

$$P_1(x) = \sum_{K=0}^{1} x_K L_{1,K}(t) = = x_0 L_{1,0}(t) + x_1 L_{1,1}(t) \qquad (5.5)$$

$$L_{1,0}(t) = \frac{t - t_1}{t_0 - t_1} \; ; \; L_{1,1}(t) = \frac{t - t_0}{t_1 - t_0}$$

The Lagrange parabolic interpolating polynomial passing trough 3 points $(t_0, x_0), (t_1, x_1)$ and (t_2, x_2) is

$$P_2(t) = \sum_{K=0}^{2} x_K L_{2,K}(t)$$

$$L_{2,0}(t) = \frac{(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)} \; L_{2,1}(t) = \frac{(t - t_0)(t - t_2)}{(t_1 - t_0)(t_1 - t_2)} \; L_{2,2}(t) = \frac{(t - t_0)(t - t_1)}{(t_2 - t_0)(t_2 - t_1)}$$

For equally spaced nodes with t t_0=0, t_1=1 and t_2=2

$$L_{2,0}(t) = \frac{(t-1)(t-2)}{2} ; \; L_{2,1}(t) = -t(t-2) ; \; L_{2,2}(t) = \frac{t(t-1)}{2}$$

$$P_2(t) = y_0 \frac{(t-1)(t-2)}{2} - y_1 t(t-2) + y_2 \frac{t(t-1)}{2}$$

Error of approximation

$$\varepsilon(t) = x_c(t) - P_{N,K}(t)$$

Sampling intervals T_0 for N=0; 1; 2 are defined as following.

N=0 – staircase approximation see (Figure 5.13)

$$T_0 = \frac{\varepsilon}{M_1}; \qquad (5.7)$$

N=1 – Linear interpolation (see Figure 5.14)

$$T_1 = \sqrt{\frac{8\varepsilon}{M_2}}; \; M_2 = |x''(t)| \qquad (5.8)$$

N=2 – Parabolic interpolation

$$T_2 = \sqrt[3]{\frac{15{,}6}{M_3}}; \; M_3 = |x'''(t)| \qquad (5.9)$$

where M_1, M_2 and M_3 are the 1st, 2nd and 3rd order derivatives absolute values.

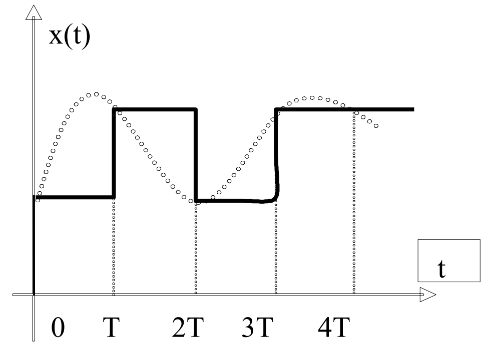


Figure 5.13

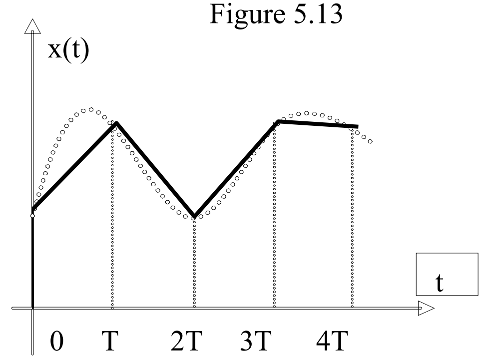


Figure 5.14

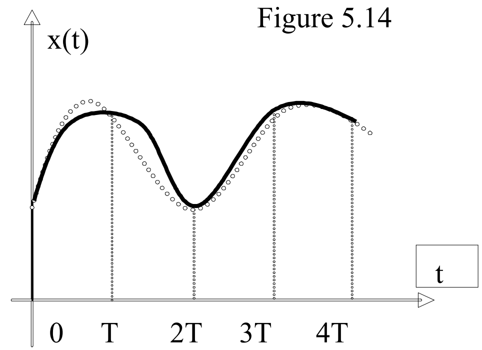


Figure 5.15

Example 5.4. Using the Matlab files perform staircase and linear approximation of y=sin(t) for t = 0: 2π.

In Figure 5.16 are shown staircase (a) and linear (b) interpolations of the sinusoidal signal using the Matlab files.

```
» t=0:pi/100:2*pi; y=sin(t); plot(t,y,'k'); hold on
» t=0:pi/4:2*pi; y=sin(t); stem(t,y,'k','fill'); hold on
» stairs(t,y) set(gca,'xtick',[0.pi/4 pi/2 3pi/4 pi 5pi/4 6pi/4 7pi/4 2pi])
»t=0:2*pi/100:2*pi; y=sin(t);ti=0:pi/4:2*pi; yi=interp1(t,y,ti);plot(t,y,ti,yi)
```

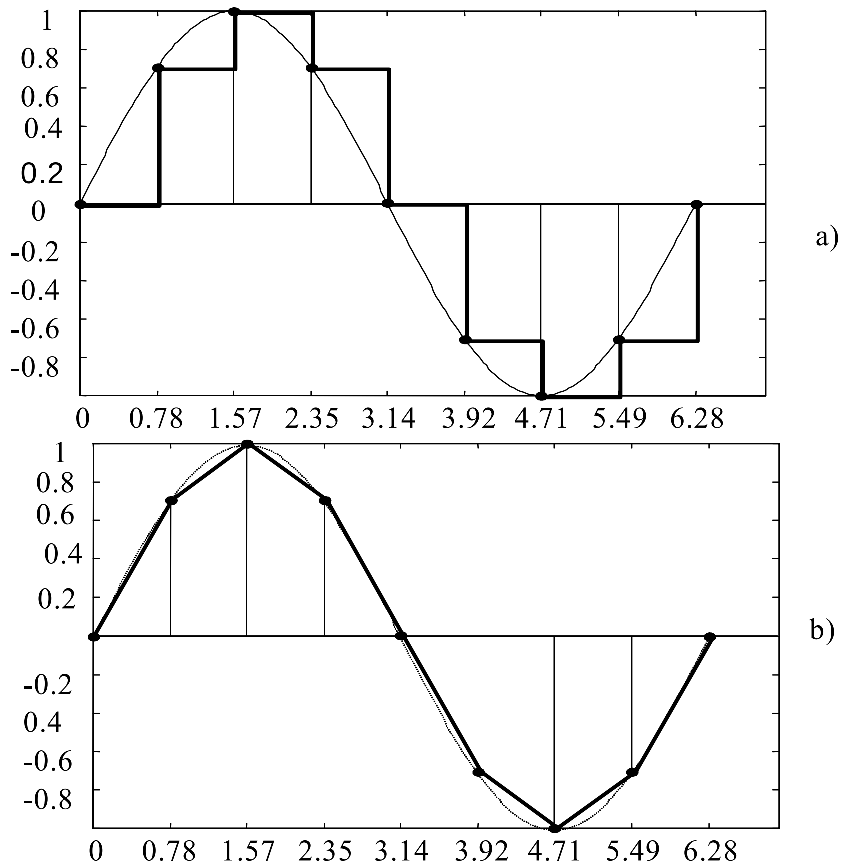


Figure 5.16